Fourier Analysis

Part I – Theory

ADRIAN CONSTANTIN

London Mathematical Society
Student Texts **85**

LONDON
MATHEMATICAL
SOCIETY
EST. 1865

CAMBRIDGE

London Mathematical Society Student Texts 85

Fourier Analysis

Part I – Theory

ADRIAN CONSTANTIN
Universität Wien, Austria

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107044104

© Adrian Constantin 2016

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2016

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloguing in Publication data
Names: Constantin, Adrian.
Title: Fourier analysis / Adrian Constantin, Universität Wien, Austria.
Description: Cambridge : Cambridge University Press, 2016- |
Series: London Mathematical Society student texts ; 85 |
Includes bibliographical references and index.
Identifiers: LCCN 2016001236 |
ISBN 9781107044104 (hardback : alk. paper) |
ISBN 9781107620353 (pbk. : alk. paper) Subjects: LCSH: Fourier analysis. |
Mathematical analysis. Classification: LCC QA403.5 .C66 2016 |
DDC 515/.2433–dc23 LC record
available at http://lccn.loc.gov/2016001236

ISBN 978-1-107-04410-4 Hardback
ISBN 978-1-107-62035-3 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party Internet Web sites referred to in this publication, and does not guarantee that any content on such Web sites is, or will remain, accurate or appropriate.

LONDON MATHEMATICAL SOCIETY STUDENT TEXTS

Managing Editor: Professor D. Benson,
Department of Mathematics, University of Aberdeen, UK

47 Introductory lectures on rings and modules. JOHN A. BEACHY
48 Set theory, ANDRÁS HAJNAL & PETER HAMBURGER. Translated by ATTILA MATE
49 An introduction to K-theory for C*-algebras, M. RØRDAM, F. LARSEN & N. J. LAUSTSEN
50 A brief guide to algebraic number theory, H. P. F. SWINNERTON-DYER
51 Steps in commutative algebra: Second edition, R. Y. SHARP
52 Finite Markov chains and algorithmic applications, OLLE HÄGGSTRÖM
53 The prime number theorem, G. J. O. JAMESON
54 Topics in graph automorphisms and reconstruction, JOSEF LAURI & RAFFAELE SCAPELLATO
55 Elementary number theory, group theory and Ramanujan graphs, GIULIANA DAVIDOFF, PETER SARNAK & ALAIN VALETTE
56 Logic, induction and sets, THOMAS FORSTER
57 Introduction to Banach algebras, operators and harmonic analysis, GARTH DALES *et al*
58 Computational algebraic geometry, HAL SCHENCK
59 Frobenius algebras and 2-D topological quantum field theories, JOACHIM KOCK
60 Linear operators and linear systems, JONATHAN R. PARTINGTON
61 An introduction to noncommutative Noetherian rings: Second edition, K. R. GOODEARL & R. B. WARFIELD, JR
62 Topics from one-dimensional dynamics, KAREN M. BRUCKS & HENK BRUIN
63 Singular points of plane curves, C. T. C. WALL
64 A short course on Banach space theory, N. L. CAROTHERS
65 Elements of the representation theory of associative algebras I, IBRAHIM ASSEM, DANIEL SIMSON & ANDRZEJ SKOWROŃSKI
66 An introduction to sieve methods and their applications, ALINA CARMEN COJOCARU & M. RAM MURTY
67 Elliptic functions, J. V. ARMITAGE & W. F. EBERLEIN
68 Hyperbolic geometry from a local viewpoint, LINDA KEEN & NIKOLA LAKIC
69 Lectures on Kähler geometry, ANDREI MOROIANU
70 Dependence logic, JOUKU VÄÄNÄNEN
71 Elements of the representation theory of associative algebras II, DANIEL SIMSON & ANDRZEJ SKOWROŃSKI
72 Elements of the representation theory of associative algebras III, DANIEL SIMSON & ANDRZEJ SKOWROŃSKI
73 Groups, graphs and trees, JOHN MEIER
74 Representation theorems in Hardy spaces, JAVAD MASHREGHI
75 An introduction to the theory of graph spectra, DRAGOŠ CVETKOVIĆ, PETER ROWLINSON & SLOBODAN SIMIĆ
76 Number theory in the spirit of Liouville, KENNETH S. WILLIAMS
77 Lectures on profinite topics in group theory, BENJAMIN KLOPSCH, NIKOLAY NIKOLOV & CHRISTOPHER VOLL
78 Clifford algebras: An introduction, D. J. H. GARLING
79 Introduction to compact Riemann surfaces and dessins d'enfants, ERNESTO GIRONDO & GABINO GONZÁLEZ-DIEZ
80 The Riemann hypothesis for function fields, MACHIEL VAN FRANKENHUIJSEN
81 Number theory, Fourier analysis and geometric discrepancy, GIANCARLO TRAVAGLINI
82 Finite geometry and combinatorial applications, SIMEON BALL
83 The geometry of celestial mechanics, HANSJÖRG GEIGES
84 Random graphs, geometry and asymptotic structure, MICHAEL KRIVELEVICH *et al*
85 Fourier analysis: Part I – Theory, ADRIAN CONSTANTIN
86 Dispersive partial differential equations, M. BURAK ERDOĞAN & NIKOLAOS TZIRAKIS

To my family

此为试读，需要完整PDF请访问：www.ertongbook.com

Preface

Fourier analysis is a central area of modern mathematics, comprising deep results that rely on advanced principles, as well as numerous aspects that require manipulative ingenuity. The power of the theory is illustrated by its wide applicability. Ideas originating in Fourier analysis permeate many essential developments of modern mathematics, bridging analysis with algebra and providing effective tools for an astonishing variety of applications. We list here an alphabetical sample of subjects, illustrating either areas of mathematics with strong links to Fourier analysis or real world applications of Fourier analysis, that are covered briefly in this textbook:[1] acoustics, complex analysis, functional analysis/operator theory, group theory/representation theory, heat flow, hydrodynamics, image processing, medical imaging, number theory, optics and astronomy, partial differential equations, probability and statistics, quantum mechanics, signal processing.

While formal approaches to Fourier analysis can be informative, to appreciate the subject fully and to strengthen the ability to use it in other contexts, one has to acquire a certain mathematical sophistication that draws on measure theory and functional analysis. Lebesgue's integral and the concepts of Hilbert and Banach spaces are intimately connected to Fourier analysis, providing not only an adequate setting but also being useful in obtaining fundamental results, often with surprisingly little effort. A detailed presentation of measure theory and functional analysis would be out of place in an introductory textbook, but

[1] The list is not exhaustive, being only indicative of the relevance of Fourier analysis to pure and applied mathematics. Part I of the textbook covers the theoretical background of Fourier analysis, while Part II is devoted to applications. Since it is impossible to draw a clear dividing line between Fourier theory and applications – for example, these aspects strongly overlap and intertwine in our discussion (in Part II) of the discrete Fourier transform and of the uncertainty principle – the difference between the material presented in the two volumes is more a question of emphasis.

ignoring these topics would amount to a lamentable attempt to run before we have learned to walk. For this reason we outline in Chapters 2 and 3 the principal facts about the Lebesgue integral and Hilbert/Banach spaces, as needed later, emphasising and illustrating the relevant conceptual ideas. This should provide some essential intuition that must, nevertheless, be adequately backed up by analytic rigour, so that we present at least sketchy proofs, avoiding only the proofs that demand an advanced degree of technical versatility. The reader may take on faith the results stated without proof, but detailed references for further study are provided. This material offers, to the interested reader, a basis for a solid grounding in these aspects and has been "class-tested" to groups of graduate students at Lund University and at the University of Vienna (during the academic years 2002–2004 and 2014–2015, respectively). However, the material in Chapters 2 and 3 is not an integral part of a standard course. Depending on the predilections of the lecturer, an all-encompassing or a minimalist point of view can be adopted. In the latter case, one can dispense altogether with measure-theoretic issues and work with the Lebesgue integral as if it were a Riemann integral, with the added bonus of Fatou's lemma and the monotone and dominated convergence theorems. The completeness, separability and density results for the spaces of integrable and square integrable functions can be taken for granted. Chapters 4–6 represent the core of the theory underpinning Fourier analysis, with various applications presented in Part II. Some applications are aimed at pure mathematicians, while others illustrate the relevance of Fourier analysis to physics and engineering. Each application was selected by virtue of its relevance and interest, but each is self-contained: the formulation of the problem is accessible and a full solution is presented. We avoid topics that can be covered only in part within a first course on Fourier analysis. The even distribution of pure and applied topics aims to cater to both mathematical backgrounds. Realistically, only about a third to a half of the applications presented in Part II can be covered in a lecture course. The available flexibility in the specific choice permits a suitable mix of pure and applied topics – the separation of pure and applied topics being, in the long run, detrimental to both areas. Whether the entire Chapter 6 belongs to the basic material on Fourier analysis is a matter of personal opinion, and thus open to debate. Parts of it could be viewed as optional reading material. Chapter 7 contains various selected advanced topics in Fourier analysis, illustrating some of the main directions in which the subject has developed. The material in Chapters 4–6, with the exception of the aspects related to distribution theory in Chapter 6, has been taught by the author as a one-semester course at King's College London during the academic years 2012–2014, while the distribution-theory aspects are an outgrowth of a lecture course on this topic at Trinity College

Dublin during 2008. In recent decades, Fourier analysis has known a period of intense technical and conceptual development which has led to a bewildering array of related topics. Nevertheless, there are a relatively small number of concepts that are commonly regarded as the bare essentials in the theory of Fourier analysis. A minimal list that could constitute a short, introductory course consists of: Section 1.1, Section 3.2, Sections 4.1–4.3, Sections 5.1–5.3, and Exercises 1.1, 3.3, 3.23, 4.1, 4.2, 4.4, 4.5, 4.11, 4.16, 4.19, 5.2, 5.5, 5.9, 5.10, 5.15, 5.16, 5.18. Each chapter is denoted by a numeral (for example, Chapter 5). The first section of the sixth chapter is denoted Section 6.1, and its second subsection is 6.1.2. Theorem 5.3 refers to the third theorem in Chapter 5 (without specification of the section or subsection), while Exercise 5.4 refers to the fourth exercise in Chapter 5. However, within Chapter 5, the 5 may be dropped and Exercise 4 used instead of Exercise 5.4.

The prerequisites are a thorough knowledge of advanced calculus and linear algebra. A large number of exercises are provided, ranging from easy to very hard, and these are supplied separately with hints and full solutions. The exercises are to be regarded as an integral part of the text, and the provided hints and solutions offer flexibility in calibrating the scale of the undertaking – in a tour it is better to admire the scenery at ease rather than to keep on schedule. Whenever the reader struggles with solving an exercise, it is worthwhile glancing at the hint prior to going through the solution that is available. Even if this proves insufficient, it might offer some valuable insight. We strive throughout for a somewhat detailed presentation, at the risk of boring those able to proceed faster. Such readers have the option of judicious skipping. We have tried to prevent our natural fondness for simplicity turning into an excuse to avoid difficulties at any price; quite often, certain difficulties are apparently circumvented rather than escaped altogether. Eventually, they may be encountered again, when they have multiplied, become more involved and hidden in a confusion of detail, which has been generated by lots of misdirected industry. On the other hand, we have tried to steer away from emphasising matters of pure technique – it is all too common to see failures of insight hidden under a blanket of excess technical detail, and focus on detail often leads to a narrowing of perspective. We hope to have found an acceptable balance between doing too little and attempting to do too much.

An appendix with some brief historical notes illustrates the international character of the underlying research efforts, being also indicative of the time needed for the crystallisation of specific concepts and ideas, as well as of their lasting value. There is a similarity between the struggle of early research mathematicians who developed and formalised a topic and the challenges embraced as one embarks upon the study of the topic.

In writing this textbook the author has acted primarily as a reporter, not a researcher: nearly all the results can be found in earlier books or in research publications. We try to offer a coherent exposition, arranging separate topics into a unified whole, and occasionally incorporating some recent developments. While we attempt to give credit where it is due, we also found that this is sometimes difficult or impossible and, as a result, in some instances, secondary sources have prevailed. The reading of parts of this book would be, we believe, beneficial during, or as a preparation for, a graduate school in mathematics – at least, the author wishes he had this material before beginning his own graduate studies.

I owe a debt of gratitude to Roger Astley of Cambridge University Press who encouraged this project from the beginning, being patient and understanding beyond the call of duty. I would like to thank the reviewers of an early and incomplete draft of the book for their constructive suggestions, which I have attempted to incorporate. I am grateful to several mathematicians for reading and commenting on the manuscript and for trying out parts of it on their classes. I cannot name one without naming them all, so they shall remain unnamed to avoid offence to those whose names have escaped me as I attempted to draw up a tentative list of acknowledgements, but they are all deeply appreciated. Despite their best efforts, there are very likely undetected errors that are my sole responsibility and for which I ask the reader to accept my apologies.

Contents

此为试读,需要完整PDF请访问: www.ertongbook.com

1
Introduction

The aim of this chapter is to introduce the concept of Fourier series in an accessible way. We present the analytic setting in which Fourier series arise as the natural generalisation of trigonometric polynomials. We also describe how the problem of the vibrating string and the investigation of heat flow mark the beginning of the theory of Fourier series as a useful approach for solving differential equations of physical relevance. A link between trigonometric polynomials and number theory is also explored.

1.1 Trigonometric polynomials and series

A *trigonometric polynomial* of degree n is an expression of the form

$$p(t) = \sum_{k=-n}^{n} c_k \, e^{2\pi ikt} \tag{1.1}$$

where the c_ks are complex numbers with $|c_{-n}| + |c_n| \neq 0$. Thus p_n is a continuous periodic function of the real variable t, of period 1, determined by its values on $[0, 1)$, or any other interval of length 1. Since

$$\int_0^1 e^{2\pi ikt}\, dt = \begin{cases} 0 & \text{if} \quad k \neq 0, \\ 1 & \text{if} \quad k = 0, \end{cases} \tag{1.2}$$

the constants c_k in the representation (1.1) of the trigonometric polynomial p can be computed by means of

$$c_k = \int_0^1 p(t)\, e^{-2\pi ikt}\, dt, \qquad |k| \leq n. \tag{1.3}$$

The function $e_k(t) = e^{2\pi ikt}$ is sometimes referred to as the *character with frequency k* or as the kth *pure frequency*.

The trigonometric polynomials (1.1) can also be looked at geometrically. Namely, we can interpret the complex number $p(t)$ in (1.1) as the vector sum of its components, each complex number c_k being modified by a supplementary phase $2\pi kt$. In the case of real positive coefficients the visual approach is particularly simple: $p(t)$ is the extremity of a polygonal contour formed by successive straight segments with respective lengths c_k, each one making the same angle $2\pi t$ with the preceding (and following) one. A simple example is depicted in Figure 1.1; for more elaborate examples we refer to the discussion in Lévy-Leblond (1997).

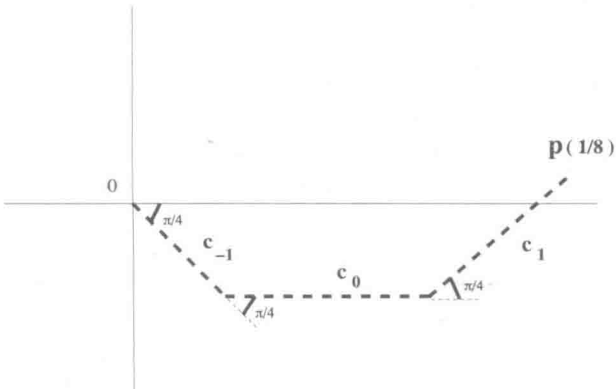

Figure 1.1 The geometric representation of the value at $t = 1/8$ of a trigonometric polynomial $p(t)$ of degree 1 and with positive coefficients.

A fundamental approximation result (to be proved in Chapter 4) is that for any continuous periodic function $f : \mathbb{R} \to \mathbb{C}$ of period 1, given $\varepsilon > 0$, there is a trigonometric polynomial p with

$$|p(t) - f(t)| < \varepsilon, \qquad t \in \mathbb{R}. \tag{1.4}$$

Due to periodicity, it suffices to verify the above inequality for $t \in [0, 1)$.

The role of the multiplicative factor (2π) in the argument of the fundamental trigonometric monomials $e^{2\pi ikt}$ used in (1.1) is to normalise the period to 1. However, given that (1.1) can be expressed as $p(t) = \sum_{k=0}^{n} a_k \cos(2\pi kt) + \sum_{k=0}^{n} b_k \sin(2\pi kt)$, for some $a_k, b_k \in \mathbb{C}$, it is reasonable to wonder why we de not associate the terminology "trigonometric polynomial" with functions of the form

$$q(t) = \sum_{k=0}^{n} \alpha_k \cos^k(2\pi t) + \sum_{k=0}^{n} \beta_k \sin^k(2\pi t) \tag{1.5}$$

for some $a_k, b_k \in \mathbb{C}$. An exercise in trigonometric identities[1] shows that any function of type (1.5)

[1] In this context, it is comforting to know that, see Borzellino and Sherman (2012), polynomial relations between $\cos(2\pi t)$ and $\sin(2\pi t)$ are always consequences of the Pythagorean identity $\cos^2(2\pi t) + \sin^2(2\pi t) = 1$; there are no hidden tricks.

can be written in the form (1.1), with the same value of the degree. However, not all trigonometric polynomials are expressible in the form (1.5): for example, $t \mapsto \sin(4N\pi t)$ with $N \neq 0$ integer are not expressible, see Borzellino and Sherman (2012). For this reason,[2] expressions of the form (1.5) are not enough to approximate well continuous periodic functions of period 1.

The approximation result expressed by means of (1.4) leads us naturally to the concept of a *trigonometric series* or *Fourier series*, defined in analogy to (1.1) as an expression of the form

$$\sum_{k=-\infty}^{\infty} c_k \, e^{2\pi ikt}, \tag{1.6}$$

representing formally a function f of period 1. In light of (1.3), we expect that the constants c_k in (1.6) and the function f are connected by the formula

$$c_k = \int_0^1 f(t)\, e^{-2\pi ikt}\, dt, \qquad k \in \mathbb{Z}. \tag{1.7}$$

More generally, the Fourier series associated to a function $f : \mathbb{R} \to \mathbb{C}$ of period $T > 0$ is

$$\sum_{k=-\infty}^{\infty} c_k \, e^{2\pi ikt/T}, \tag{1.8}$$

where

$$c_k = \frac{1}{T} \int_0^T f(t)\, e^{-2\pi ikt/T}\, dt, \qquad k \in \mathbb{Z}. \tag{1.9}$$

The theory of Fourier series studies the classes of periodic functions (of period $T > 0$) and the notions of convergence appropriate for the correspondence $f(t) \approx \sum_{k=-\infty}^{\infty} c_k \, e^{2\pi ikt/T}$, with the constants c_k given by (1.9), expressing the function f in terms of a superposition of oscillations with frequencies $\nu_k = k/T$ that are integer multiples of the fundamental frequency $\nu = 1/T$. As a glimpse at the intricacy of the subject, notice that above we pointed out that for any continuous periodic function $f : \mathbb{R} \to \mathbb{C}$ of period 1 we can find trigonometric polynomials that approximate it uniformly, that is, in the sense of (1.4). Nevertheless, the specific trigonometric polynomials obtained by means of the symmetric partial sums

$$s_n(f,t) = \sum_{k=-n}^{n} c_k \, e^{2\pi ikt} \tag{1.10}$$

with c_k given by (1.9) are not necessarily good approximations: the sequence

[2] The orthogonality considerations made in Chapter 3 show that if we rely only on functions of the form (1.5), then the approximations miss out an infinite-dimensional subspace of the space of square integrable functions.

$\{s_n(f,t)\}_{n\geq 1}$ might diverge for infinitely many values of $t \in [0,1]$; see the discussion in the introduction to Chapter 4. This shows that continuity coupled with the concept of pointwise or uniform convergence is not adequate. The proper setting turns out to be the class of Lebesgue integrable or square integrable functions, with an associated concept of convergence. The need to go beyond the class of continuous functions and the classical theory of Riemann integrable functions is fully justified by the mathematical power and flexibility of the theory within the new setting, and is further emphasised by its wide range of applicability.

1.2 The dawn of the theory

Fourier analysis dates back to late eighteenth/early nineteenth century studies of the vibrating string and of heat propagation. Two basic partial differential equations of one-dimensional mathematical physics are the wave equation

$$\frac{\partial^2 U}{\partial T^2} = c^2\, \frac{\partial^2 U}{\partial X^2} \tag{1.11}$$

and the heat equation

$$\frac{\partial U}{\partial T} = \kappa\, \frac{\partial^2 U}{\partial X^2}\,, \tag{1.12}$$

where $c > 0$ and $\kappa > 0$ are physical constants. In (1.11), $U = U(X,T)$ represents, at the location X and at time T, the displacement of a homogeneous string placed in the (X,Y)-plane and stretched along the X-axis between $X = 0$ and $X = L$, where it is tied. The value of the constant c is $\sqrt{\tau/\rho}$, where τ is the tension coefficient of the string and ρ is its mass density. Equation (1.11) is to be solved for $T > 0$ and X between 0 and L, subject to the boundary conditions

$$U(0,T) = U(L,T) = 0, \qquad T \geq 0, \tag{1.13}$$

which express the fact that the endpoints of the string are fixed. The solution U describes the vibrations of a violin string. On the other hand, in (1.12), $U = U(X,T)$ is the temperature in a homogeneous, straight wire of length L, whose endpoints are held at constant temperature zero. The value of the constant κ in (1.12) is specific to the conducting material. The problem is to describe the temperature at time T from its knowledge at time $T = 0$. Consequently, we seek solutions to (1.12) for $T > 0$ and X between 0 and L, subject to the boundary conditions

$$U(0,T) = U(L,T) = 0, \qquad T \geq 0, \tag{1.14}$$

此为试读,需要完整PDF请访问: www.ertongbook.com