BOUNDED ANALYTIC FUNCTIONS

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ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers
New York London Toronto Sydney San Francisco

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Garnett, John B.
Bounded analytic functions.

Bibliography: p.
Includes index.
1. Analytic functions. 2. Functional analysis.
I. Title.
QA331.G32 515.7 80-26298
ISBN 0-12-276150-2

PRINTED IN THE UNITED STATES OF AMERICA

81 82 83 84 9 8 7 6 5 4 3 2 1

Preface

The primary purpose of this book is to teach technique, and we have emphasized method rather than generality. Many of the ideas we shall introduce, from subharmonicity and maximal functions to Littlewood–Paley integrals, Carleson measures, and stopping time constructions, extend naturally to Euclidean space and beyond; but for unity and simplicity we have limited their discussion to one dimension. Some of these ideas are explored more fully in the books of Stein [1970] and Stein and Weiss [1971].

Our secondary purpose is to give a self-contained view of the contemporary theory of bounded analytic functions on the unit disc. To do that we must treat in detail certain notions, such as conformal invariance, the subharmonicity of $\log|f|$, dual extremal problems, and, especially, Blaschke products, which do not yet generalize well from their classical setting. Readers interested in higher dimensions or in multiply connected domains are only advised that the proofs in the text most resisting generalization are those relying on Blaschke products or dual extremal problems. Freeing certain single-variable proofs from these notions is tantamount to solving some of today's most difficult problems on the unit ball of $\mathbb{C}^n$.

On the other hand, readers patient with one complex variable will be rewarded with a theory richer in texture. For example, a basic question about the conjugation operator leads to functions of bounded mean oscillation, which leads to Carleson measures, and in turn, via Blaschke products or duality, to interpolating sequences and the corona theorem. Only the last link of the chain does not generalize. The proof of the H^1–BMO duality and the construction behind the corona theorem, both amenable to higher dimensions, merge to yield a remarkable characterization, in terms of Blaschke products, of the closed algebras between H^∞ and L^∞.

This book presents a particular viewpoint, both in method and material; it is no encyclopedia. Some topics, such as interpolation problems and the arguments behind the corona theorem, have been pursued at length, while other topics, such as the applications of Banach algebra theory and the vast interaction between H^∞ and operator theory, have been minimized. (For the connections with operator theory, we recommend the excellent works of Douglas [1972] and of Sarason [1979].) Whenever possible we have used conformal invariance and real variables techniques. Over the past twenty years, the renewed interest in H^∞ has been prompted largely by functional analytic questions, but I believe that solving some of the harder problems now facing the subject requires returning to the disc or the circle and formulating more constructive arguments.

Prerequisites for this book are basic courses in real and complex analysis; the first eleven chapters of Rudin's textbook [1974] should be sufficient. In Chapter I we present some additional background not usually found in elementary graduate courses. Chapters II–V form an introduction to Hardy space theory, through conjugate functions, dual extremal problems, and some of the uniform algebra aspects of H^∞. We have based the theory on maximal functions and on subharmonicity. (For different approaches, see the books of Hoffman [1962a] and Duren [1970].) Chapters VI–X develop the ideas surrounding the John–Nirenberg theorem, the geometry of interpolating sequences, and the corona theorem. People already familiar with the field will notice that these chapters largely grew out of two papers by Carleson [1958, 1962a]. Much of the material in the last five chapters has not appeared in monograph form before. The book is self-contained, and the first half is basically a preparation for the second half. However, the early sections of Chapters VI–VIII contain essential parts of today's classical H^p theory, while a few specialized items have infiltrated Chapter IV. The notes by Koosis [1980] provide a more elementary and less intense survey of some of the topics we have considered.

Results are numbered lexicographically within each chapter, so that "Theorem 1.3" is the third item of Section 1 of the same chapter, whereas "Theorem 1.3 of Chapter I" or "Theorem I.1.3" is in Section 1 of Chapter I. Independently, the same convention is used to number formulas, such as "(1.10) of Chapter III."

There are 31 figures in the text. Understand the figures and you understand the book.

Each chapter ends with some bibliographical notes and a section called "Exercises and Further Results." Some exercises are intended for beginners, while others, the "further results," are theorems not in the text. They usually include references, which serve also to suggest that they may not be elementary. Sometimes extensive hints have been given, and occasionally an exercise with thorough hints is referred to later in the text. Especially satisfying exercises have been marked with one, two, or three stars ★.

Acknowledgments

Without the help and encouragement of many friends this book would never have been finished. I am fortunate to have had two students, Donald Marshall and Peter Jones, each of whom has turned this subject in his own direction. I thank them for their numerous improvements on the text and for the mathematics they have taught me. Over the years I have found the advice and encouragement from Ted Gamelin, Paul Koosis, and Nicholas Varopoulos most heartening. Irving Glicksberg, still my teacher, bombarded me with corrections, mathematical and stylistic. Were it not for him, this book would have been a great deal harder to read. Others who have provided valuable assistance with the subject or the text include Anthony Carbery, Lennart Carleson, David Drasin, John Fagarason, Michael Frazier, Gregory Gibbons, Leslie Kay, Steven Krantz, Robert Latter, Robyn Owens, Donald Sarason, and Allen Shields. The manuscript, including myriad scribbled revisions, was typed by Debra Remetch, who was aided by Sarah Remetch. I thank the staff of Academic Press for their patience during production.

I am very grateful to the University of California, Los Angeles, with its pleasant atmosphere, efficient staff, congenial colleagues, and stimulating students, for giving me considerable professional liberty. Much of this book was written while I was *professeur associé* at Université de Paris-Sud, which provided me with an excellent library, exciting seminars, and even greater professional liberty. I thank its *équipe d'analyse harmonique* for a most pleasant year.

List of Symbols

$\bar{z}$ = complex conjugate of z

$\bar{f}$ = complex conjugate of f

$\bar{\mathfrak{F}} = \{\bar{f} : f \in \mathfrak{F}\}$, when $\mathfrak{F}$ is a set of functions

$\bar{E}$ = closure of E, when E is a point set

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I

Preliminaries

As a preparation, we discuss three topics from elementary real or complex analysis which will be used throughout this book.

The first topic is the invariant form of Schwarz's lemma. It gives rise to the pseudohyperbolic metric, which is an appropriate metric for the study of bounded analytic functions. To illustrate the power of the Schwarz lemma, we prove Pick's theorem on the finite interpolation problem

$$f(z_j) = w_j, \qquad j = 1, 2, \ldots, n,$$

with $|f(z)| \leq 1$.

The second topic is from real analysis. It is the circle of ideas relating Poisson integrals to maximal functions.

The chapter ends with a brief introduction to subharmonic functions and harmonic majorants, our third topic.

1. Schwarz's Lemma

Let D be the unit disc $\{z : |z| < 1\}$ in the complex plane and let $\mathscr{B}$ denote the set of analytic functions from D into $\bar{D}$. Thus $|f(z)| \leq 1$ if $f \in \mathscr{B}$. The simple but surprisingly powerful Schwarz lemma is this:

Lemma 1.1. *If $f(z) \in \mathscr{B}$, and if $f(0) = 0$, then*

$$\begin{aligned} &|f(z)| \leq |z|, \qquad z \neq 0, \\ &|f'(0)| \leq 1. \end{aligned} \tag{1.1}$$

Equality holds in (1.1) at some point z if and only if $f(z) = e^{i\varphi}z$, φ a real constant.

The proof consists in observing that the analytic function $g(z) = f(z)/z$ satisfies $|g| \leq 1$ by virtue of the maximum principle

We shall use the invariant form of Schwarz's lemma due to Pick. A *Möbius transformation* is a conformal self-map of the unit disc. Every Möbius transformation can be written as

$$\tau(z) = e^{i\varphi}\frac{z - z_0}{1 - \bar{z}_0 z}$$

with φ real and $|z_0| < 1$. With this notation we have displayed $z_0 = \tau^{-1}(0)$.

Lemma 1.2. *If $f(z) \in \mathscr{B}$, then*

$$\frac{|f(z) - f(z_0)|}{|1 - \overline{f(z_0)}f(z)|} \le \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|, \qquad z \ne z_0, \tag{1.2}$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}. \tag{1.3}$$

Equality holds at some point z if and only if $f(z)$ is a Möbius transformation.

The proof is the same as the proof of Schwarz's lemma if we regard $\tau(z)$ as the independent variable and

$$\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}$$

as the analytic function. Letting z tend to z_0 in (1.2) gives (1.3) at $z = z_0$, an arbitrary point of D.

The *pseudohyperbolic distance* on D is defined by

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|.$$

Lemma 1.2 says that analytic mappings from D to D are Lipschitz continuous in the pseudohyperbolic distance:

$$\rho(f(z), f(w)) \le \rho(z, w).$$

The lemma also says that the distance $\rho(z, w)$ is invariant under Möbius transformations:

$$\rho(z, w) = \rho(\tau(z), \tau(w)).$$

We write $K(z_0, r)$ for the noneuclidean disc

$$K(z_0, r) = \{z : \rho(z, z_0) < r\}, \qquad 0 < r < 1.$$

Since the family $\mathscr{B}$ is invariant under the Möbius transformations, the study of the restrictions to $K(z_0, r)$ of functions in $\mathscr{B}$ is the same as the study of their restrictions to $K(0, r) = \{|w| < r\}$. In such a study, however, we must give $K(z_0, r)$ the coordinate function $w = \tau(z) = (z - z_0)/(1 - \bar{z}_0 z)$.

For example, the set of derivatives of functions in $\mathscr{B}$ do not form a conformally invariant family, but the expression

$$|f'(z)|(1 - |z|^2) \tag{1.4}$$

is conformally invariant. The proof of this fact uses the important identity

$$1 - \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|^2 = \frac{(1 - |z|^2)(1 - |z_0|^2)}{|1 - \bar{z}_0 z|^2} = (1 - |z|^2)|\tau'(z)|, \tag{1.5}$$

which is (1.3) with equality for $f(z) = \tau(z)$. Hence if $f(z) = g(\tau(z)) = g(w)$, then

$$|f'(z)|(1 - |z|^2) = |g'(w)||\tau'(z)|(1 - |z|^2) = |g'(w)|(1 - |w|^2)$$

and this is what is meant by the invariance of (1.4).

The noneuclidean disc $K(z_0, r)$, $0 < r < 1$, is the inverse image of the disc $|w| < r$ under

$$w = \tau(z) = \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Consequently $K(z_0, r)$ is also a euclidean disc $\Delta(c, R) = \{z : |z - c| < R\}$, and as such it has center

$$c = \frac{1 - r^2}{1 - r^2|z_0|^2} z_0 \tag{1.6}$$

and radius

$$R = r\frac{1 - |z_0|^2}{1 - r^2|z_0|^2}. \tag{1.7}$$

These can be found by direct calculation, but we shall derive them geometrically. The straight line through 0 and z_0 is invariant under τ, so that $\partial K(z_0, r) = \tau^{-1}(|w| = r)$ is a circle orthogonal to this line. A diameter of $K(z_0, r)$ is therefore the inverse image of the segment $[-rz_0/|z_0|, rz_0/|z_0|]$. Since $z = (w + z_0)/(1 + \bar{z}_0 w)$, this diameter is the segment

$$[\alpha, \beta] = \left[\frac{|z_0| - r}{1 - r|z_0|}\frac{z_0}{|z_0|}, \frac{|z_0| + r}{1 + r|z_0|}\frac{z_0}{|z_0|}\right]. \tag{1.8}$$

The endpoints of (1.8) are the points of $\overline{K(z_0, r)}$ of largest and smallest modulus. Thus $c = (\alpha + \beta)/2$ and $R = (|\beta| - |\alpha|)/2$ and (1.6) and (1.7) hold. Note that if r is fixed and if $|z_0| \to 1$, then the euclidean radius of $K(z_0, r)$ is asymptotic to $1 - |z_0|$.

Corollary 1.3. *If $f(z) \in \mathscr{B}$, then*

$$|f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}. \tag{1.9}$$

Proof. By Lemma 1.2, $\rho(f(z), f(0)) \leq |z|$, so that $f(z) \in \overline{K(f(0), |z|)}$. The bound on $|f(z)|$ then follows from (1.8). Equality can hold in (1.9) only if f is a Möbius transformation and $\arg z = \arg f(0)$ when $f(0) \neq 0$. □

The pseudohyperbolic distance is a metric on D. The triangle inequality for ρ follows from

Lemma 1.4. *For any three points z_0, z_1, z_2 in D,*

$$\frac{\rho(z_0, z_2) - \rho(z_2, z_1)}{1 - \rho(z_0, z_2)\rho(z_2, z_1)} \leq \rho(z_0, z_1) \leq \frac{\rho(z_0, z_2) + \rho(z_2, z_1)}{1 + \rho(z_0, z_2)\rho(z_2, z_1)}. \tag{1.10}$$

Proof. We can suppose $z_2 = 0$ because ρ is invariant. Then (1.10) become

$$\frac{|z_0| - |z_1|}{1 - |z_0||z_1|} \leq \left| \frac{z_1 - z_0}{1 - \bar{z}_0 z_1} \right| \leq \frac{|z_0| + |z_1|}{1 + |z_0||z_1|}. \tag{1.11}$$

If $|z_1| = r$, then $z = (z_1 - z_0)/(1 - \bar{z}_0 z_1)$ lies on the boundary of the noı euclidean disc $K(-z_0, r)$. On this disc $|z|$ lies between the moduli of tł endpoints of the segment (1.8). That proves (1.11). Of course (1.10) an especially (1.11) are easy to verify directly. □

Every Möbius transformation $w(z)$ sending z_0 to w_0 can be written

$$\frac{w - w_0}{1 - \bar{w}_0 w} = e^{i\varphi} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Differentiation then gives

$$|w'(z_0)| = \frac{1 - |w_0|^2}{1 - |z_0|^2}. \tag{1.12}$$

This identity we have already encountered as (1.3) with equality. By (1.12) the expression

$$ds = \frac{2|dz|}{1 - |z|^2} \tag{1.13}$$

is a conformal invariant of the disc. We can use (1.13) to define the hyperbolic length of a rectifiable arc γ in D as

$$\int_\gamma \frac{2|dz|}{1 - |z|^2}.$$

We can then define the *Poincaré metric* $\psi(z_1, z_2)$ as the infimum of the hyperbolic lengths of the arcs in D joining z_1 to z_2. The distance $\psi(z_1, z_2)$ is then conformally invariant. If $z_1 = 0$, $z_2 = r > 0$, it is not difficult to see that

$$\psi(z_1, z_2) = 2\int_0^r \frac{dx}{1 - |x|^2} = \log\frac{1 + r}{1 - r}.$$

Since any pair of points z_1 and z_2 can be mapped to 0 and $\rho(z_1, z_2) = |(z_2 - z_1)/(1 - \bar{z}_1 z_2)|$, respectively, by a Möbius transformation, we therefore have

$$\psi(z_1, z_2) = \log\frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)}.$$

A calculation then gives

$$\rho(z_1, z_2) = \frac{\tanh \psi(z_1, z_2)}{2}.$$

Moreover, because the shortest path from 0 to r is the radius, the geodesics, or paths of shortest distance, in the Poincaré metric consist of the images of the diameter under all Möbius transformations. These are the diameters of D and the circular arcs in D orthogonal to ∂D. If these arcs are called lines, we have a model of the hyperbolic geometry of Lobachevsky.

In this book we shall work with the pseudohyperbolic metric ρ rather than with ψ, although the geodesics are often lurking in our intuition.

Hyperbolic geometry is somewhat simpler in the upper half plane $\mathscr{H} = \{z = x + iy : y > 0\}$. In $\mathscr{H}$

$$\rho(z_1, z_2) = \left|\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right|$$

and the element of hyperbolic arc length is

$$ds = \frac{|dz|}{y}.$$

Geodesics are vertical lines and circles orthogonal to the real axis. The conformal self-maps of $\mathscr{H}$ that fix the point at ∞ have a very simple form:

$$\tau(z) = az + x_0, \qquad a > 0, \quad x_0 \in \mathbb{R}.$$

Horizontal lines $\{y = y_0\}$ can be mapped to one another by these self-maps of $\mathscr{H}$. This is not the case in D with the circles $\{|z| = r\}$ in D. In $\mathscr{H}$ any two squares

$$\{x_0 < x < x_0 + h, h < y < 2h\}$$

are congruent in the noneuclidean geometry. The corresponding congruent figures in D are more complicated. For these and for other reasons, $\mathscr{H}$ is often the more convenient domain for many problems.

2. Pick's Theorem

A *finite Blaschke product* is a function of the form

$$B(z) = e^{i\varphi}\prod_{j=1}^{n}\frac{z - z_j}{1 - \bar{z}_j z}, \qquad |z_j| < 1.$$

The function B has the properties

(i) B is continuous across ∂D,
(ii) $|B| = 1$ on ∂D, and
(iii) B has finitely many zeros in D.

These properties determine B up to a constant factor of modulus one. Indeed, if an analytic function $f(z)$ has (i)–(iii), and if $B(z)$ is the finite Blaschke product with the same zeros, then by the maximum principle, $|f/B| \le 1$ and $|B/f| \le 1$, on D, and so f/B is constant. The *degree* of B is its number of zeros. A Blaschke product of degree 0 is a constant function of absolute value 1.

Theorem 2.1 (Carathéodory). *If $f(z) \in \mathscr{B}$, then there is a sequence $\{B_k\}$ of finite Blaschke products that converges to $f(z)$ pointwise on D*

Proof. Write

$$f(z) = c_0 + c_1 z + \cdots.$$

By induction, we shall find a Blaschke product of degree at most n whose first n coefficients match those of f;

$$B_n = c_0 + c_1 z + \cdots + c_{n-1}z^{n-1} + d_n z^n + \cdots.$$

That will prove the theorem. Since $|c_0| \le 1$, we can take

$$B_0 = \frac{z + c_0}{1 + \bar{c}_0 z}.$$

If $|c_0| = 1$, then $B_0 = c_0$ is a Blaschke product of degree 0. Suppose that for each $g \in \mathscr{B}$ we have constructed $B_{n-1}(z)$. Set

$$g = \frac{1}{z}\frac{f - f(0)}{1 - \overline{f(0)}f}$$

and let B_{n-1} be a Blaschke product of degree at most $n - 1$ such that $g - B_{n-1}$ has $n - 1$ zeros at 0. Then $zg - zB_{n-1}$ has n zeros at $z = 0$. Set

$$B_n(z) = \frac{zB_{n-1}(z) + f(0)}{1 + \overline{f(0)}zB_{n-1}(z)}.$$