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Observables and Symmetries of n-Plectic Manifolds

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BestMasters
ISBN 978-3-658-12389-5 ISBN 978-3-658-12390-1 (eBook)
DOI 10.1007/978-3-658-12390-1

Library of Congress Control Number: 2015960848

Springer Spektrum

Printed on acid-free paper

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(www.springer.com)

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Acknowledgements

I would like to express my deepest appreciation to my advisor Professor Dr. Tilmann Wurzbacher, who has been an ideal advisor for various reasons. Not only has he proposed the highly interesting topic of the thesis and given me optimal preparation to attacking it through the courses he offered during the last few years, but he also invested a tremendous amount of time in guiding and supporting me in the course of the composition of this thesis. Without his patience, mathematical (and didactic) skill I would have never been able to master this thesis. I would also like to thank him for his enthusiasm and sense of humour which provided "a light in the dark" during the more technical parts of work.

I would like to express my gratefulness to the Gerhard C. Starck Stiftung for the financial and moral support during my studies. Thanks for giving me the possibility to fully concentrate on the subject, that I am so passionate about, and for the exceptional people that I had the privilege to meet through the Stiftung.

I would also like to thank my family and Leonie, who have been a great emotional and practical support during the course of the thesis, who had to endure a lot of vacuum, where I should have been in the last few months. Their care and sympathy were of great value for me.

Last but not least I would like to thank my friends, who have turned the past 4 years of studies into an unforgettable time. It is thanks to them, that studying was fun from the first day to the last. Special thanks goes to Stefan Heuver, who has shown great understanding and kindness during the last few months. Thanks for his great advices, as for instance "If you worry, just stop worrying". Thanks for showing me that my problems were not all that bad, by explaining his problems and last but not least, thanks for forcing me to have a little break once in a while.

Leonid Ryvkin

Contents

1 Introduction

The aim of this thesis is the analysis and understanding of n-plectic (also called multisymplectic) manifolds, their observables and their symmetries.

The development of a geometric theory of n-plectic manifolds naturally starts with the classical case of symplectic manifolds. In that setting one considers a manifold M equipped with a non-degenerate closed two-form ω. This two-form relates $C^\infty(M)$, the space of smooth functions on M with the vector fields on M. This relation induces a Lie algebra structure on $C^\infty(M)$, which, endowed with this structure, is called the Lie algebra of observables of M. The infinitesimal version ζ of a symplectic group action on (M, ω) can often be linearly lifted to this Lie algebra. Such group actions are called weakly Hamiltonian and are the basis of what might be called "momentum geometry", which is also of great interest for physics. The existence of an equivariant moment map is obstructed by a certain cohomology class.

In the general n-plectic case we consider a manifold M with a closed, non-degenerate $n+1$-form. Unlike the symplectic (1-plectic) case, the non-degeneracy of ω only guarantees injectiveness of the map $v \mapsto \iota_v\omega$ but no surjectiveness. Also the space of observables carries a more general algebraic structure which turns it into a "Lie n-algebra", which is a Lie algebra only when $n = 1$.

Lie n-algebras are a special case of L_∞-algebras, whose study constitutes the first chapter of this thesis. Lie n-algebras from n-plectic manifolds are then introduced in Chapter two. The generalization of the notion of a (co-)moment map is the main theme of the third chapter, culminating in an explicit calculation of the obstruction classes to strongly Hamiltonian n-plectic group actions.

The thesis is structured in the following way:

- Sections 1.1 and 1.2 develop two possible perspectives on L_∞-algebras. The earlier one describes an L_∞-algebra as a $\mathbb{Z}$-graded vector space, with a family of brackets $\{l_k | 1 \leq k\}$ satisfying an infinite series of multi-bracket identites. The latter standpoint views an L_∞-algebra as a graded vector space V endowed with a degree one coderivation $D : S^\bullet(sV) \to S^\bullet(sV)$, which squares to zero, where $S^\bullet(sV)$ denotes the cofree graded commutative coalgebra of the vector space sV, where sV is V with degrees shifted by 1. The main result of these sections, the equivalence of these descriptions (Theorem 2.14) is based on the proof sketched in [7], but we work out the full details of the proof here.
- Section 1.3 then discusses the right notion of morphisms of L_∞-algebras. Though L_∞-morphisms more tangible from the coalgebra standpoint, the multi-bracket formulation is more practical for calculations. Hence, the main result of this section is the multi-bracket formulation of L_∞-morphisms (Lemma 2.18). It is based on the calculations made in Appendix A of [5] and results in a formula already stated in in [1], but without proof. We giva a detailed proof and we are also more explicit about the precise range of the summations involved.

- Section 1.4 gives an introduction to the representation theory of L_∞-algebras according to [6]. It presents two perspectives on representations: via L_∞-morphisms into the endomorphism space of a differential graded vector space and via L_∞-modules. The equivalence of both approaches is only quoted from [6] in this thesis.

- Section 1.5 then reduces the "heavy machinery" developed in the preceeding sections to the case of gounded Lie m-algebras, which are of special interest as they include the L_∞-algebras formed by the observables of a multisymplectic manifold. This section mostly reproduces results from [5], where the property "to be grounded" is referred to as "having Property (P)".

- Section 2.1 introduces the notion of n-plectic (or multisymplectic) manifolds and derives their basic properties in accordance with [11]. The discussion of the pre-n-plectic case mostly relies on [5]. The Lie n-algebra of observables is described for both cases.

- Sections 3.1 and 3.2 develop the theory of Hamiltonian group actions on n-plectic manifolds, guided by the classical case described e.g. in [9]. The (homotopy) co-moment map which is the basic object of our discussions was first introduced in [5]. The main results of these sections are Lemma 4.7, which describes the obstruction for an n-plectic action to be weakly Hamiltonian and Theorem 4.13, the which gives a cohomological description of the obstruction for a weakly Hamiltonian action to be strongly Hamiltonian. The latter is a refinement of Theorem 9.7 in [5]. Furthermore we show that the first resp. the last component of a strong homotopy co-moment map yields a covariant multimoment map in the sense of [3] resp. a multi-moment map in the sense of [8].

2 L_∞-algebras

Let $\mathbb{K}$ denote here a fixed ground field of characteristic 0. All vector spaces, linear maps and tensor products will be defined with respect to/taken over this field, unless we explicitly state otherwise. A good overview of the subject of L_∞-algebras is provided in the n-lab ([12]).

2.1 Generalizing differential graded Lie algebras

In this section we will define L_∞-algebras as the generalization of Lie algebras. For doing so let us first review the notion of a Lie algebra: A Lie algebra is a vector space L with a skew-symmetric bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying the Jacobi identity:

$$[[x_1, x_2], x_3] - [[x_1, x_3], x_2] + [[x_2, x_3], x_1] = 0 \tag{1}$$

This can be rewritten in the following way, where $P = \{\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)\} \subset S_3$:

$$\sum_{\sigma \in P} sgn(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0.$$

The set P consists precisely of those permutations, which move one element to the last position, without distorting the inner order of the others. As we will deal with identities of multi-brackets soon, we will have to generalize this notion of "moving one element out of three to the end" to "moving q elements out of $p + q$ to the end". The permutations doing that are exactly the (p, q)-unshuffles.

Definition 2.1. A permutation $\sigma \in S_{p+q}$ is a *(p, q)-unshuffle* if and only if $\sigma(i) < \sigma(i + 1)$ for $i \neq p$. We denote the set of of (p, q)-unshuffles by $ush(p, q) \subset S_{p+q}$.

The condition in the above definition guarantees that the first p and the last q elements stay in the same internal order. These permutations are called unshuffles, because their inverses correspond to shuffling a deck of p cards into a deck of q cards.

Let us now turn to the graded context. First we introduce a grading on our vector space: We set $L = \bigoplus_{i \in \mathbb{Z}} L_i$, where L_i ist the vector subspace of elements of degree i. We will write $|x| = i$ if $x \in L_i$. A Lie structure $[\cdot,\cdot]$ on such a vector space L should satisfy the following three conditions:

- $[L_i, L_j] \subset L_{i+j}$ (the bracket respects the grading)
- $[x_1, x_2] = -(-1)^{|x_1||x_2|}[x_2, x_1]$ for all homogenous $x_1, x_2 \in L$ (the bracket is graded skew-symmetric)
- $(-1)^{|x_1||x_3|}[x_1, [x_2, x_3]] + (-1)^{|x_1||x_2|}[x_2, [x_3, x_1]] + (-1)^{|x_2||x_3|}[x_3, [x_1, x_2]] = 0$ for all homogenous elements $x_1, x_2, x_3 \in L$ (the graded Jacobi identity holds)

If we try to bring the graded Jacobi identity into form of equation (1), we get:

$$[[x_1, x_2], x_3] - (-1)^{|x_2||x_3|}[[x_1, x_3], x_2] + (-1)^{|x_1||x_2|+|x_1||x_3|}[[x_2, x_3], x_1] = 0.$$

So $[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}]$ gets an additional sign for every transposition of two odd elements. This leads us to the definition of the Koszul sign ϵ of a permutation σ acting on elements $x_1, ..., x_n \in L$.

Definition 2.2. Let $\sigma \in S_n$ be a permutation acting on elements $v_1, ..., v_n$ of a $\mathbb{Z}$-graded vector space V. Let $(v_{i_1}, ..., v_{i_k})$ be the ordered sublist of $v_1, ..., v_n$ including exactly the odd elements. Then there is a (unique) permutation $\tilde{\sigma} \in S_k$ such that $(v_{i_{\tilde{\sigma}(1)}}, ..., v_{i_{\tilde{\sigma}(k)}})$ is the ordered sublist of $v_{\sigma(1)}, ..., v_{\sigma(n)}$ including exactly the odd elements. Then the *Koszul sign of σ acting on $v_1, ..., v_n$* is defined by

$$\epsilon(\sigma, v_1, ..., v_n) := sgn(\tilde{\sigma}).$$

Remark 2.3. One can check that ϵ is well-behaved in the sense that

$$\epsilon(\sigma' \circ \sigma, v_1, ..., v_k) = \epsilon(\sigma', v_{\sigma(1)}, ..., v_{\sigma(k)}) \cdot \epsilon(\sigma, v_1, ..., v_k),$$

and that for a transposition τ_i interchanging v_i and v_{i+1} it holds that $\epsilon(\tau_i, v_1, ..., v_n) = (-1)^{|v_i||v_{i+1}|}$.

Thus, the graded Jacobi identity can be written in the following way:

$$\sum_{\sigma \in ush(2,1)} sgn(\sigma)\epsilon(\sigma, x_1, x_2, x_3)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0 \tag{2}$$

Next we consider a graded vector space with a differential $d : L \to L$ satisfying $d(L_i) \subset L_{i+1}$ and $d^2 = 0$. This turns L into a differential graded vector space or, in other words, into a chain complex[1]:

$$\cdots \longrightarrow L_{i-2} \xrightarrow{d} L_{i-1} \xrightarrow{d} L_i \xrightarrow{d} L_{i+1} \xrightarrow{d} \cdots$$

Adopting the standard language, used e.g. for de Rham cohomology, we call $x \in L$ *closed* if $dx = d(x) = 0$ and *exact* if $x = dy$ for some $y \in L$. In the latter case y is called a *potential* for x.

A differential graded vector space (L, d) together with a graded Lie bracket $[\cdot, \cdot]$ on L is called a *differential graded Lie algebra* if the differential derives the bracket i.e. satisfies the following graded Leibniz rule (for $x_1, x_2 \in L$):

$$d[x_1, x_2] = [d(x_1), x_2] - (-1)^{|x_1|}[x_1, d(x_2)].$$

This can be rewritten as:

$$d[x_1, x_2] = [d(x_1), x_2] - (-1)^{|x_1||x_2|}[d(x_2), x_1].$$

The latter equation can also be written in terms of signed sums of unshuffles. In fact, it is equivalent to:

$$\sum_{\sigma \in ush(2,0)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)d[x_{\sigma(1)}, x_{\sigma(2)}] = \sum_{\sigma \in ush(1,1)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)[d(x_{\sigma(1)}), x_{\sigma(2)}].$$

[1]From the perspective of homological algebra it would be a cochain complex, as the differential has positive degree. The closed elements would be called cocycles and the exact elements coboundaries.

If we furthermore interpret the differential as a unary bracket and write $x \mapsto [x]$ instead of $x \mapsto d(x)$ we get:

$$\sum_{\sigma \in ush(2,0)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)[[x_{\sigma(1)}, x_{\sigma(2)}]] = \sum_{\sigma \in ush(1,1)} sgn(\sigma)\epsilon(\sigma, x_1, x_2)[[x_{\sigma(1)}], x_{\sigma(2)}] \tag{3}$$

Even the condition $d^2 = 0$ can be written as $[[x]] = 0$, or bringing it into the form of the other equations :

$$\sum_{\sigma \in ush(1,0)} sgn(\sigma)\epsilon(\sigma, x_1)[[x_{\sigma(1)}]] = 0 \tag{4}$$

We have now learned how to describe a differential graded Lie algebra as a graded vector space L with a unary bracket $[\cdot]$ of degree one and a binary bracket $[\cdot,\cdot]$ of degree 0, which satisfy the equations (2), (3) and (4). These three equations are special cases of the below equation (5). We will now generalize the notion of differential graded Lie algebra to obtain a definition of an L_∞-algebra.

Definition 2.4. An *L_∞-algebra* (or *Lie-∞-algebra*) is a graded vector space $L = \bigoplus_{i\in\mathbb{Z}} L_i$ together with a family of graded skew-symmetric multilinear maps $\{l_i : \bigtimes^i L \to L \mid i \in \mathbb{N}\}$ such that l_i has degree $2-i$ and the following identity holds (for all $n \in \mathbb{N}$):

$$\sum_{i+j=n+1} (-1)^{i(j+1)} \sum_{\sigma \in ush(i,n-i)} sgn(\sigma)\epsilon(\sigma, x_1, ..., x_n)\, l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)} ..., x_{\sigma(n)}) = 0 \tag{5}$$

To keep the definition transparent despite the use of (possibly infinitely many) multi-brackets we write l_n for the n-ary bracket and give a brief overview of the multilinear algebra surpressed in the definition.

- $\bigtimes^i L$ is the cross product of i copies of L.
- A multi-linear map $l_i : \bigtimes^i L \to L$ is a map linear in every component. One could equivalently define l_i as a linear map from $\bigotimes^i L$ to L. Here $\bigotimes^i L$ is the i-fold tensor product of the graded vector space L.
- Demanding l_i to have degree $2-i$ means that l_i restricted to $L_{k_1} \times L_{k_2} \times ... \times L_{k_i}$ must map into $L_{k_1+k_2+...+k_i+2-i}$. Turning $\bigotimes^i L$ into a graded vector space by defining the degree of $x_1 \otimes ... \otimes x_i$ as $\sum_{k=1}^i |x_k|$, this translates to saying that $l_i : \bigotimes^i L \to L$ is a linear map of degree $2-i$.
- The maps l_i being (graded) skew-symmetric means that for all $\sigma \in S_i$ the identity $l_i(x_1, ..., x_i) = sgn(\sigma)\epsilon(\sigma, x_1, ..., x_i)l_i(x_{\sigma(1)}, ..., x_{\sigma(i)})$ holds. Using the language of multilinear algebra this is equvialent to $l_i : \bigotimes^i L \to L$ descending to a map $l_i : E^i(L) \to L$, where $E^i(L)$ is the i-th (graded) exterior power of L.

- The reason why we only sum over the unshuffles is to avoid repetition: If any two permutations $\sigma, \sigma' \in S_n$ differ by a permutation which only interchanges the first i and the last $n-i$ elements (i.e. $\sigma = \tau \circ \sigma'$, where $\tau = (\tau_1, \tau_2) \in S_i \times S_{n-i} \subset S_n$) then $l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}...x_{\sigma(n)}) = \pm l_j(l_i(x_{\sigma'(1)}, ..., x_{\sigma'(i)}), x_{\sigma'(i+1)}...x_{\sigma'(n)})$. Thus, up to sign, we would add up essentially same elements several times. To avoid that, we need a system of representatives for $S_n/(S_i \times S_{n-i})$. That system of representatives is provided by the Unshuffles. All inner permutations of the first i and last $n-i$ elements are prohibited, as there is only one way to arrange an i-element (resp. $(n-i)$-element) subset of $\{1, ..., n\}$ in a strictly ascending order.

Next, let us take a closer look at the signs involved:

- The first sign in (5) is $(-1)^{i(j+1)}$. It only gives us an additional sign if the number of elements consumed by brackets is odd for the inner one (l_i) and even for the outer one (l_j). If we look at (5) for $n = 2$ we get equation (3). If not for the $(-1)^{i(j+1)}$ in our definition this equation would have an additional sign and the graded Leibniz rule would not be satisfied.

- $sgn(\sigma)$ is the usual sign of the permutation, not depending on the grading. It contributes a minus sign for every transposition in the permutation.

- $\epsilon(\sigma, x_1, ..., x_i)$ is the Koszul sign, which is highly dependant on the grading of the elements permuted. It gives us a factor of -1 for every transposition of two odd elements in the permutation.

Finally, by direct calculation, we can see that this definition indeed provides a generalization of a (differential graded) Lie algebra:

Example 2.5. A differential graded Lie algebra is precisely an L_∞-algebra with $l_i = 0$ for $i > 2$. Furthermore, a Lie algebra is an L_∞-algebra where the L is concentrated in degree zero (i.e. $L = L_0$).

2.2 L_∞-algebras as coalgebras with differentials

Having defined L_∞-algebra objects, we would now like to investigate their structure-preserving maps. Intuitively one would regard (say degree-zero) linear maps f which conserve the brackets i.e. $l'_i(f(x_1), ..., f(x_i)) = f(l_i(x_1, ..., x_i))$. Unfortunately this nontion of morphism is not flexible enough, and from our definition it is not clear what the right notion of more "flexible" maps would be. In order to advance, we will reformulate our definition in the language of "differential-graded coalgebras". First of all we will get rid of the different degrees of the l_n. We define the shift sL to be L with the grading shifted by one i.e. $(sL)_i = L_{i+1}$. To keep notation consistent we denote the multi-brackets by sl_i instead of l_i if we work with the shifted grading. Then we get

$$(sl_i)((sL)_{k_1}, ..., (sL)_{k_i}) = l_i(L_{k_1+1}, ..., L_{k_i+1}) \subset L_{k_1+1+...+k_i+1+2-i} = L_{k_1+...k_i+2} = (sL)_{k_1+...+k_i+1}.$$

So $(sl_i) : \bigtimes^i(sL) \to (sL)$ have degree one independantly of i. Unfortunately the sl_i are not in general skew-symmetric with respect to the new grading. But we can turn the l_i into graded

commutative maps by altering the signs.
Recall that a multi-linear map $f : \bigtimes^n V \to W$, where V is a graded vector space and W is any vector space, is called *graded-commutative* (or *graded-symmetric*) if it satisfies

$$f(y_1, ..., y_n) = \epsilon(\sigma, y_1, ..., y_n) f(y_{\sigma(1)}, ..., y_{\sigma(n)}) \quad \forall \sigma \in S_n \tag{6}$$

Writing $sx \in (sL)_n$ for an element $x \in L_{n+1}$, we now define $\hat{l}_n : \bigtimes^n(sL) \to (sL)$ by

$$\hat{l}_n(sx_1, ..., sx_n) = (-1)^{\alpha(x_1,...,x_n)}(sl_n)(sx_1, ..., sx_n)$$

with α defined in the following way:

$$\alpha(x_1, ..., x_n) = \alpha_n(x_1, ..., x_n) = \sum_{\substack{i \text{ is odd} \\ i \in \{1,...,n\}}} |x_i| \quad \text{when n is even,}$$

$$\alpha(x_1, ..., x_n) = \alpha_n(x_1, ..., x_n) = 1 + \sum_{\substack{i \text{ is even} \\ i \in \{1,...,n\}}} |x_i| \quad \text{when n is odd.}$$

Lemma 2.6. The maps $\hat{l}_n : \bigtimes^n(sL) \to sL$ are graded-symmetric multi-linear maps of degree 1 for all $n \in \mathbb{N}$.

Proof. The $\hat{l}_n$ are multi-linear by construction and $deg(\hat{l}_n) = 1$ follows from the discussion above. The only property, which remains to be checked is the graded symmetry (6). However, it is enough to check this property for the transpositions $\{\tau_i \mid 1 \le i < n\} \subset S_n$, where τ_i interchanges y_i and y_{i+1}, as these transpositions generate the full symmetric group. So we calculate:

$$\begin{aligned} \hat{l}_n(sx_1, ..., sx_n) &= (-1)^{\alpha(x_1,...,x_n)} sl_n(sx_1, ..., sx_n) = (-1)^{\alpha(x_1,...,x_n)} l_n(x_1, ..., x_n) \\ &= (-1)^{\alpha(x_1,...,x_n)} sgn(\tau_i)\epsilon(\tau_i, x_1, ..., x_n) l_n(x_{\tau_i(1)}, ..., x_{\tau_i(n)}). \end{aligned}$$

As τ_i consists of only one transposition we have $sgn(\tau_i) = -1$ and $\epsilon(\tau_i, x_1, ..., x_n) = (-1)^{|x_i||x_{i+1}|}$, thus

$$\begin{aligned} \hat{l}_n(sx_1, ..., sx_n) &= (-1)^{\alpha(x_1,...,x_n)}(-1)(-1)^{|x_i||x_{i+1}|} l_n(x_{\tau_i(1)}, ..., x_{\tau_i(n)}) \\ &= (-1)^{\alpha(x_1,...,x_n)}(-1)(-1)^{|x_i||x_{i+1}|} sl_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}) \\ &= (-1)^{\alpha(x_1,...,x_n)}(-1)(-1)^{|x_i||x_{i+1}|}(-1)^{\alpha(x_{\tau_i(1)},...,x_{\tau_i(n)})} \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}). \end{aligned}$$

But the sum $\alpha(x_1, ..., x_n)$ and $\alpha(x_{\tau_i(1)}, ..., x_{\tau_i(n)})$ only differ by one summand. One of them has the summand $|x_i|$ and the other one $|x_{i+1}|$. All the other signs cancel out, so we get:

$$\begin{aligned} \hat{l}_n(sx_1, ..., sx_n) &= (-1)^{|x_i|+|x_j|}(-1)(-1)^{|x_i||x_{i+1}|} \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}) = (-1)^{(|x_i|+1)(|x_{i+1}|+1)} \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}) \\ &= \epsilon(\tau_i, sx_1, ..., sx_n) \hat{l}_n(sx_{\tau_i(1)}, ..., sx_{\tau_i(n)}). \end{aligned}$$

□

Remark 2.7. In this proof the constant summand in the definition of α for odd n is not needed. It will turn out useful later, when we discuss the multi-bracket equation (5).

We can now regard $\hat{l}_n$ as a graded-symmetric linear map from $\bigotimes^n(sL)$ to sL of degree one. And, analogously, we can encode the symmetry properties of this linear map by modifying its domain. This time we need the graded-symmetric powers of (sL), which are denoted by $S^n(sL)$. Analogusly to the preceeding reasoning we can describe $\hat{l}_n$ equivalently as linear maps $\hat{l}_n : S^n(sL) \to sL$, where $S^n(\cdot)$ is defined as follows:

Definition 2.8. Let V be a $\mathbb{Z}$-graded vector space. *The k-th (graded) symmetrization operator* $Sym_k : \bigotimes^k V \to \bigotimes^k V$ is defined by

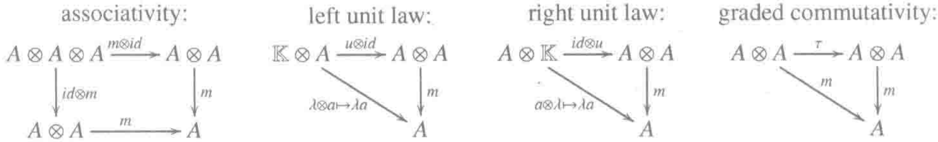
$$Sym_k(v_1 \otimes ... \otimes v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma, v_1, ..., v_k) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)}.$$

The Image of Sym_k is a vector subspace and called *the k-th (graded) symmetric power of V*. We will denote $Sym_k(v_1 \otimes ... \otimes v_k)$ by $v_1 \odot ... \odot v_k$.

The sum of the $S^n(sL)$ forms an algebra $S^\bullet(sL) = \bigoplus_{n \in \mathbb{N}_0} S^n(sL)$, the so-called *free graded-commutative algebra* on sL. Setting $\hat{l}_0 : S^0(sL) = \mathbb{K} \to (sL)$ to be the zero map, we can combine the $\hat{l}_i$ to a linear map $\hat{l} : S^\bullet(sL) \to sL$ of degree 1. This map encodes all information of our L_∞-algebra except equation (5).

To encode equation (5) we have to change perspectives. Instead of regarding $S^\bullet(sL)$ as an algebra we will use its coalgebra structure. The reason for this is that the universal property of an algebra gives us a way to extend the domain of maps $sL \to W$. What we need here is the dual property: We want to extend a map $W \to sL$ to some map $W \to S^\bullet(sL)$ (in our case $W = S^\bullet(sL)$ and the map to be extended is $\hat{l}$). The most direct way to understand a coalgebra is by dualizing the diagrams encoding an algebra structure. Let us first recall the definition of a graded algebra in terms of diagrams:

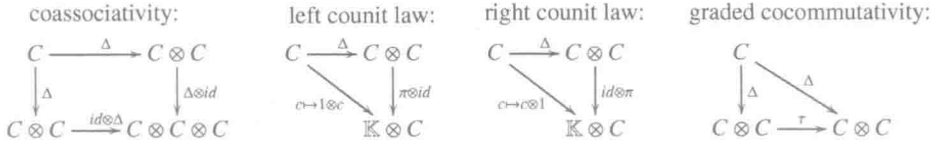
Definition 2.9. Let A be a graded vector space. A *unital associative graded commutative algebra-structure on A* consists of two linear maps of degree zero, a linear multiplication map $m : A \otimes A \to A$ and a unit morphism $u : \mathbb{K} \to A$ such that the following diagrams commute, where $\tau : A \otimes A \to A \otimes A$ is defined by $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$:



"Reversing all arrows" gives us the definition of a coalgebra:

Definition 2.10. Let C be a graded vector space. A *counital coassociative graded cocommutative coalgebra-structure on C* consists of two linear maps of degree zero, a comultiplication map

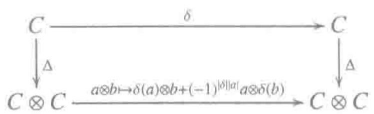
$\Delta : C \otimes C \to C$ (also called diagonal) and a counit morphism $\pi : C \to \mathbb{K}$ such that the following diagrams commute:



One can turn $S^\bullet(sL)$ into a counital coassociative graded cocommutative coalgebra with the counit $\pi = \pi_0$ given by the projection onto $S^0(sL) = \mathbb{K}$ and the following diagonal:

$$\Delta(sx_1 \odot ... \odot sx_n) := \sum_{i=0}^{n} \sum_{\sigma \in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)}\right) \otimes \left(sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}\right)$$

More generally, this construction can be carried out for any graded vector space V. The resulting coalgebra structure on the graded vector space $S^\bullet V$ is then called the *cofree graded cocommutative coalgebra* of V. For any counital coassociative algebra C a homomorphism $\delta : C \to C$ is called a *coderivation* if the following diagram commutes:



Returning to our case, we have a map $\hat{l} : S^\bullet(sL) \to sL = S^1(sL)$. It can be extended to a degree 1 coderivation $D : S^\bullet(sL) \to S^\bullet(sL)$ in the following way:

Definition 2.11. Let L be a $\mathbb{Z}$-graded vector space and $\{l_i \mid i \in \mathbb{N}\}$ a family of graded skew-symmetric maps. Let $\hat{l} : S^\bullet(sL) \to sL$ be the above defined map. Then we define the degree-one derivation $D : S^\bullet(sL) \to S^\bullet(sL)$ by the following formula:

$$D(sx_1 \odot ... \odot sx_n) = \sum_{i=1}^{n} \sum_{\sigma \in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(\hat{l}(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)}.$$

Remark 2.12. As we defined $\hat{l}$ summand-wise by the $\hat{l}_i$, we could instead write

$$D(sx_1 \odot ... \odot sx_n) = \sum_{i=1}^{n} \sum_{\sigma \in ush(i,n-i)} \epsilon(\sigma, sx_1, ..., sx_n) \left(\hat{l}_i(sx_{\sigma(1)} \odot ... \odot sx_{\sigma(i)})\right) \odot sx_{\sigma(i+1)} \odot ... \odot sx_{\sigma(n)},$$

and check that D indeed is a coderivation. As a concequence of corollary A.2 D is the unique coderivation satisfying $\pi_1 D = \hat{l}$.

With this extension done, we can finally reformulate condition (5).

Lemma 2.13. In the setting of the last definition the equation

$$\sum_{i+j=n+1}(-1)^{i(j+1)}\sum_{\sigma\in ush(i,n-i)} sgn(\sigma)\epsilon(\sigma,x_1,...,x_n)\; l_j(l_i(x_{\sigma(1)},...,x_{\sigma(i)}),x_{\sigma(i+1)}...,x_{\sigma(n)})=0$$

implies $D^2=0$.

Proof.

$$D^2(sx_1\odot...\odot sx_n)=D\left(\sum_{i=1}^{n}\sum_{\sigma\in ush(i,n-i)}\epsilon(\sigma,sx_1,...,sx_n)\left(\hat{l}_i(sx_{\sigma(1)}\odot...\odot sx_{\sigma(i)})\right)\odot sx_{\sigma(i+1)}\odot...\odot sx_{\sigma(n)}\right)$$
$$=\sum_{i=1}^{n}\sum_{\sigma\in ush(i,n-i)}\epsilon(\sigma,sx_1,...,sx_n)D\left(\left(\hat{l}_i(sx_{\sigma(1)}\odot...\odot sx_{\sigma(i)})\right)\odot sx_{\sigma(i+1)}\odot...\odot sx_{\sigma(n)}\right)$$

Next we apply D to a term of length $n-i+1$. For that we will define $y_1^\sigma:=\hat{l}_i(sx_{\sigma(1)}\odot...\odot sx_{\sigma(i)})$ and $y_{k+1}^\sigma:=sx_{\sigma(i+k)}$ for $k\in\{1,...,n-i\}$. So we have:

$$D\left(\left(\hat{l}_i(sx_{\sigma(1)}\odot...\odot sx_{\sigma(i)})\right)\odot sx_{\sigma(i+1)}\odot...\odot sx_{\sigma(n)}\right)=D(y_1^\sigma\odot...\odot y_{n-i+1}^\sigma)$$
$$=\sum_{j=1}^{n-i+1}\sum_{\tau\in ush(j,n-i+1-j)}\epsilon(\tau,y_1^\sigma,...,y_{n-i+1}^\sigma)\hat{l}_j(y_{\tau(1)}^\sigma\odot...\odot y_{\tau(j)}^\sigma)\odot y_{\tau(j+1)}^\sigma\odot...\odot y_{\tau(n-i+1)}^\sigma.$$

As y_1^σ is a structurally different term than the other y_k^σ we will distinguish between τ satisfying $\tau(1)=1$ denoted by $u_1(j,n,i)$ and $\tau(j+1)=1$ denoted by $u_2(j,n,i)$. Every element in $ush(j,n-i+1-j)$ is exactly in one of those two subsets. Let us first analyse the case $\tau\in u_2(j,n,i)$:

$$\hat{l}_j(y_{\tau(1)}^\sigma\odot...\odot y_{\tau(j)}^\sigma)\odot y_{\tau(j+1)}^\sigma\odot...\odot y_{\tau(n-i+1)}^\sigma$$
$$=\hat{l}_j(y_{\tau(1)}^\sigma\odot...\odot y_{\tau(j)}^\sigma)\odot\hat{l}_i(sx_{\sigma(1)}\odot...\odot sx_{\sigma(i)})\odot y_{\tau(j+2)}^\sigma\odot...\odot y_{\tau(n-i+1)}^\sigma.$$

The total sign of this element is $\epsilon(\sigma,sx_1,...,sx_n)\cdot\epsilon(\tau,y_1^\sigma,...,y_{n-i+1}^\sigma)$. Now we regard the summand coming from $\tilde{i}=j,\tilde{j}=i$ and $\tilde{\sigma}\in ush(\tilde{i},n-\tilde{i})$, $\tilde{\tau}\in u_2(\tilde{j},n,\tilde{i})$ defined as follows:

- $\tilde{\sigma}$ is the unshuffle that, given the strictly ascending list $(sx_1,...,sx_n)$, moves the elements $y_{\tau(1)}^\sigma,...,y_{\tau(j)}^\sigma$ to the front.
- $\tilde{\tau}$ is the unshuffle that, given a list starting with $\hat{l}_{\tilde{i}}(y_{\tau(1)}^\sigma\odot...\odot y_{\tau(j)}^\sigma)$ and continuing with a strictly ascending list of $\left\{y_{\tau(j+2)}^\sigma,...,y_{\tau(n-i+1)}^\sigma\right\}\cup\left\{sx_{\sigma(1)},...,sx_{\sigma(i)}\right\}$, moves the elements $(sx_{\sigma(1)},...,sx_{\sigma(i)})$ to the front.

By construction the summands belonging to (i,j,σ,τ) and $(\tilde{i},\tilde{j},\tilde{\sigma},\tau)$ are equal up to sign. The signs come from the odd transpositions in $\sigma,\tau,\tilde{\sigma},\tilde{\tau}$ and from interchanging $\hat{l}_i(sx_{\sigma(1)}\odot...\odot sx_{\sigma(i)})$ with $\hat{l}_j(y_{\tau(1)}^\sigma\odot...\odot y_{\tau(j)}^\sigma)$.

First of all let us take a look at the transpositions nessecary to move the elements $y_{\tau(k)}^\sigma$ with