Quantum Electrodynamics

SURAJ N. GUPTA

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Distinguished Professor of Physics
Wayne State University, Detroit, Michigan

GORDON AND BREACH SCIENCE PUBLISHERS
New York Paris London

Gordon and Breach Science Publishers, Inc.
One Park Avenue
New York, NY10016

Gordon and Breach Science Publishers Ltd.
42 William IV Street
London WC2N 4DF

Gordon & Breach
7–9 rue Emile Dubois
Paris 75014

Library of Congress Cataloging in Publication Data

Gupta, Suraj N 1924–
Quantum electrodynamics.

Includes index.
1. Quantum electrodynamics. I. Title.
QC680.G86 537.6 77-73566
ISBN 0-677-04240-X

Preface

Although our full understanding of elementary particle physics remains an exciting and elusive goal, there is much in quantum electrodynamics that is considered well established. A comprehensive knowledge of quantum electrodynamics is indispensable for theoretical as well as experimental physicists in the exploration of the interactions of elementary particles. I have aimed at a concise and coherent presentation of the basic theory, calculational techniques, and important applications of quantum electrodynamics. Mathematical manipulations are explained in sufficient detail, and many derivations are provided in unusually simplified form.

In order to clarify the underlying ideas, fundamental principles of the quantum theory of fields are fully discussed, and the general formalism is developed in a manner that is also applicable to a considerable extent to strong, weak, and gravitational interactions.

This book is essentially self-contained, but it presupposes knowledge acquired in a standard one-year graduate level quantum mechanics course. I have made special effort to ensure that the book is useful and rewarding to both researchers and students.

It is a pleasure to thank Sandra Hoffmann for secretarial help in the preparation of the manuscript.

Suraj N. Gupta

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CHAPTER 1

Classical Theory of Fields

It is well known that classical mechanics is able to explain only a limited type of experimental results, while more refined experiments have revealed the need for quantum mechanics. Although the classical theory does not provide us with a complete description of nature, it is nevertheless a self-consistent theory. It is simpler and more familiar than the quantum theory, and is able to explain the observed facts within its limited domain. Moreover, in the formulation of the quantum theory we are often guided by the classical theory. It will, therefore, be useful to be familiar with the classical electromagnetic field and some general principles of the classical field theory.

1 Tensors and Lorentz Transformations

We shall first explain the tensor notation to be used in this book. A four-vector will be denoted as A_μ and a tensor of the second rank as $A_{\mu\nu}$, where Greek indices always take the values 1, 2, 3, 4. In particular, the four-dimensional coordinates of a point will be denoted as x_μ, where x_1, x_2, and x_3 are the space coordinates, and $x_4 = ict$. The space components of any four-vector A_μ will be denoted as A_i or **A**, where Latin indices always take the values 1, 2, 3. A repeated Greek or Latin index will imply summation over all values of the index. We shall often explicitly use the index 0 instead of 4, the connection between the two indices being

$$A_4 = iA_0, \qquad A_{i4} = iA_{i0}, \qquad A_{41} = iA_{0i}, \qquad A_{44} = i^2 A_{00} = -A_{00}, \tag{1.1}$$

but it should be noted that we shall denote $\partial/\partial x_\mu$ as ∂_μ and $\partial/\partial x_0$ as ∂_0, and consequently $\partial_4 = -i\partial_0$. For convenience we shall sometimes drop the indices of four-vectors, denoting A_μ as A, $A_\mu B_\mu$ as $A \cdot B$, A_μ^2 as A^2, and ∂_μ^2 as ∂^2. We shall also denote $dx_1\,dx_2\,dx_3$ as $d\mathbf{x}$ and $dx_1\,dx_2\,dx_3\,dx_0$ as dx.

A four-vector A_μ will be said to be real if A_i and A_0 are real, and similarly a tensor $A_{\mu\nu}$ will be said to be real if A_{ik}, A_{i0}, A_{0i}, and A_{00} are real. Special care is required in dealing with complex conjugates of tensors. We shall always use an asterisk to denote the complex conjugate of any quantity that is not a tensor. An asterisk will also denote the complex conjugate of any tensor component that carries only the indices 1, 2, 3, 0, but if the index 4 occurs n times in a tensor component, then an asterisk will denote $(-1)^n$ times the complex conjugate of the component. Thus, the complex conjugate of the relation $A_4 = iA_0$ is

$$-A_4^* = -iA_0^*$$

or

$$A_4^* = iA_0^*, \tag{1.2}$$

and similarly

$$A_{i4}^* = iA_{i0}^*, \qquad A_{4i}^* = iA_{0i}^*, \qquad A_{44}^* = -A_{00}^*. \tag{1.3}$$

It follows that if A_μ and $A_{\mu\nu}$ are real, then

$$A_\mu^* = A_\mu, \qquad A_{\mu\nu}^* = A_{\mu\nu}. \tag{1.4}$$

Any linear transformation of the space-time coordinates x_μ that leaves the quantity x_μ^2 invariant is called a Lorentz transformation, and such a transformation is expressible as

$$x'_\mu = a_{\mu\nu} x_\nu \tag{1.5}$$

with

$$a_{\mu\lambda} a_{\nu\lambda} = a_{\lambda\mu} a_{\lambda\nu} = \delta_{\mu\nu}. \tag{1.6}$$

All Lorentz transformations are made up of one or more of the following three transformations: (1) proper Lorentz transformations, (2) space inversion, and (3) time reversal. A proper Lorentz transformation corresponds to a continuous rotation of the space-time axes, while under space inversion

$$x'_i = -x_i, \qquad x'_4 = x_4, \tag{1.7}$$

and under time reversal

$$x'_i = x_i \qquad x'_4 = -x_4. \tag{1.8}$$

Space inversion is also known as the parity operation, and a quantity is said to have even or odd parity according as it remains unchanged or undergoes a change of sign under space inversion.

Let us consider a Lorentz transformation that corresponds to an infinitesimal rotation of the space-time axes. Under such a transformation

$$x'_\mu = x_\mu + \omega_{\mu\nu} x_\nu, \tag{1.9}$$

where the $\omega_{\mu\nu}$ are infinitesimal quantities, so that

$$a_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu},$$

which, when substituted into (1.6), gives

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{1.10}$$

It is also useful to note that

$$\partial\omega_{\mu\nu}/\partial\omega_{\alpha\beta} = \delta_{\mu\alpha}\,\delta_{\nu\beta} - \delta_{\mu\beta}\,\delta_{\nu\alpha}. \tag{1.11}$$

Any proper Lorentz transformation can be built up from the simpler infinitesimal transformations given by (1.9) and 1.10).

2 Laws of Classical Electrodynamics

Classical electrodynamics deals with the interaction of the electromagnetic field with charged material bodies. The electromagnetic field is described by the electric field strength **E** and the magnetic field strength **H**, which are functions of space and time. On the other hand, the charged material bodies give rise to a charge density ρ and a current density $\rho\mathbf{v}$ in the field, where $\mathbf{v}$ is the velocity of the charge. The dependence of the electromagnetic field on the presence of the charges is given by the Maxwell equations

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\rho\mathbf{v}, \tag{2.1}$$

$$\nabla \cdot \mathbf{E} = \rho, \tag{2.2}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0, \tag{2.3}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{2.4}$$

while the force experienced by the charges due to the electromagnetic field is given by the Lorentz equation

$$\mathbf{K} = \rho\mathbf{E} + \frac{\rho}{c}(\mathbf{v} \times \mathbf{H}), \tag{2.5}$$

where **K** denotes the density of force exerted by the electromagnetic field on the charges.

The above equations can be expressed in a covariant form, if we regard $\rho\mathbf{v}$ and $c\rho$ as the components of a four-vector j_μ, and **E** and **H** as the components of an antisymmetrical tensor $F_{\mu\nu}$, such that

$$(j_1, j_2, j_3) = \rho\mathbf{v}, \qquad j_4 = ic\rho, \tag{2.6}$$

$$(F_{23}, F_{31}, F_{12}) = \mathbf{H}, \qquad (F_{41}, F_{42}, F_{43}) = i\mathbf{E}. \tag{2.7}$$

We can then express (2.1) and (2.2) as

$$\partial_\nu F_{\mu\nu} = \frac{1}{c} j_\mu, \tag{2.8}$$

and (2.3) and (2.4) as

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0. \tag{2.9}$$

It is also possible to write (2.5) in the covariant form

$$K_\mu = \frac{1}{c} F_{\mu\nu} j_\nu, \tag{2.10}$$

where the space components of K_μ correspond to $\mathbf{K}$. In order to interpret the meaning of the time component of K_μ, we note that, according to (2.10), (2.6), and (2.7),

$$K_0 = \frac{1}{c} F_{0i} j_i = \frac{\rho}{c} (\mathbf{E} \cdot \mathbf{v}),$$

which gives, on account of (2.5),

$$K_0 = \frac{1}{c} (\mathbf{K} \cdot \mathbf{v}) = \frac{w}{c}, \tag{2.11}$$

where $w = \mathbf{K} \cdot \mathbf{v}$ denotes the rate of work performed by the field on the charges per unit volume.

Differentiation of (2.8) with respect to x_μ yields

$$\partial_\mu \partial_\nu F_{\mu\nu} = \frac{1}{c} \partial_\mu j_\mu.$$

Since $F_{\mu\nu}$ is antisymmetrical, the left side of the above equation vanishes, and thus

$$\partial_\mu j_\mu = 0, \tag{2.12}$$

which expresses the conservation of charge.

3 Electromagnetic Energy-Momentum Tensor

It is possible to express the force density (2.10) as the four-divergence of a tensor by using the field equations (2.8) and (2.9) and the fact that $F_{\mu\nu}$ is antisymmetrical. For, according to (2.8) and (2.10),

$$K_\mu = F_{\mu\nu} \partial_\rho F_{\nu\rho} = \partial_\rho (F_{\mu\nu} F_{\nu\rho}) - F_{\nu\rho} \partial_\rho F_{\mu\nu}, \tag{3.1}$$

while multiplication of (2.9) by $F_{\nu\rho}$ gives

$$F_{\nu\rho}\partial_\rho F_{\mu\nu} + F_{\nu\rho}\partial_\nu F_{\rho\mu} + F_{\nu\rho}\partial_\mu F_{\nu\rho} = 0 \tag{3.2}$$

or, with interchange of the indices ν and ρ in the second term of (3.2),

$$2F_{\nu\rho}\partial_\rho F_{\mu\nu} + F_{\nu\rho}\partial_\mu F_{\nu\rho} = 0,$$

so that

$$F_{\nu\rho}\partial_\rho F_{\mu\nu} = -\tfrac{1}{2}F_{\nu\rho}\partial_\mu F_{\nu\rho} = -\tfrac{1}{4}\partial_\mu F_{\nu\rho}^2. \tag{3.3}$$

It then follows from (3.1) and (3.3) that

$$K_\mu = \partial_\rho(F_{\mu\nu}F_{\nu\rho}) + \tfrac{1}{4}\partial_\mu(F_{\nu\rho}^2),$$

which can be put in the form

$$K_\mu = -\partial_\nu T_{\mu\nu} \tag{3.4}$$

with

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu\rho} - \tfrac{1}{4}\delta_{\mu\nu}F_{\lambda\rho}^2. \tag{3.5}$$

The tensor $T_{\mu\nu}$, which is symmetrical, is called the energy-momentum tensor of the electromagnetic field.

In order to interpret the physical meaning of the various components of $T_{\mu\nu}$, we write (3.4) as

$$K_\mu = -\partial_i T_{\mu i} - \frac{1}{c}\frac{\partial T_{\mu 0}}{\partial t},$$

so that, with integration over a volume V bounded by a surface S,

$$\int K_\mu \, d\mathbf{x} = -\int \partial_i T_{\mu i}\, d\mathbf{x} - \frac{1}{c}\frac{d}{dt}\int T_{\mu 0}\, d\mathbf{x}.$$

The first term on the right side of the above equation can be converted into a surface integral, which gives

$$\int K_\mu \, d\mathbf{x} = -\int T_{\mu i}\, dS_i - \frac{1}{c}\frac{d}{dt}\int T_{\mu 0}\, d\mathbf{x}, \tag{3.6}$$

where dS_i is an element of surface multiplied by a unit vector normal to the surface element in the outward direction.

Let us consider the time component of (3.6), which is given by

$$\int K_0 \, d\mathbf{x} = -\int T_{0i}\, dS_i - \frac{1}{c}\frac{d}{dt}\int T_{00}\, d\mathbf{x}$$

or, in view of (2.11),

$$-\frac{d}{dt}\int T_{00}\, d\mathbf{x} = \int cT_{0i}\, dS_i + \int w \, d\mathbf{x}. \tag{3.7}$$

The above equation expresses the law of conservation of energy, if we regard $\int T_{00}\, d\mathbf{x}$ as the electromagnetic energy in the volume V, and $\int cT_{0i}\, dS_i$ as the rate of flow of energy out of the surface S. For, then (3.7) simply means that the rate of decrease of energy in the volume V is equal to the rate of flow of energy out of the volume V plus the rate of work performed by the field within this volume. Thus, the energy density H in the electromagnetic field is

$$H = T_{00} = F_{0\rho} F_{0\rho} + \tfrac{1}{4}F_{\lambda\rho}^2, \tag{3.8}$$

and the rate of flow of energy in any direction through unit area is

$$N_i = cT_{0i} = cF_{0\mu} F_{i\mu}, \tag{3.9}$$

the vector N_i being called the Poynting vector. These results can also be expressed in terms of the electric and magnetic field strengths with the use of (2.7) as

$$H = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2), \qquad \mathbf{N} = c(\mathbf{E} \times \mathbf{H}). \tag{3.10}$$

Similarly, we consider the space components of (3.6), which are given by

$$\int K_i\, d\mathbf{x} = -\int T_{ik}\, dS_k - \frac{1}{c}\frac{d}{dt}\int T_{i0}\, d\mathbf{x}$$

or

$$-\frac{d}{dt}\int \frac{1}{c} T_{i0}\, d\mathbf{x} = \int T_{ik}\, dS_k + \int K_i\, d\mathbf{x}. \tag{3.11}$$

The above equation expresses the law of conservation of momentum, if we regard $\int (T_{i0}/c)\, d\mathbf{x}$ as the momentum of the electromagnetic field within the volume V, and $\int T_{ik}\, dS_k$ as the rate of flow of momentum out of this volume. For, then (3.11) means that the rate of decrease of momentum in the volume V is equal to the rate of flow of momentum out of the volume V plus the force exerted by the field within this volume. It follows that in an electromagnetic field the momentum density G_i is

$$G_i = \frac{1}{c} T_{i0}, \tag{3.12}$$

and the rate of flow of the ith component of momentum along the x_k axis through unit area is T_{ik}.

It is interesting that although the Lorentz equation for the force density is a basic equation of electrodynamics, the sign of the force density is fixed by the Maxwell equations. For, if we change the sign of the right side of (2.10), the corresponding energy-momentum tensor will differ from (3.5) by a negative sign. Thus, the energy density in the electromagnetic field will have the negative value $-\tfrac{1}{2}(\mathbf{E}^2 + \mathbf{H}^2)$, which is physically inadmissible.

4 Electromagnetic Potential

In order to express the Maxwell equations in a simpler form, we put

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.1}$$

where the four-vector A_μ is called the electromagnetic potential. Since the tensor $F_{\mu\nu}$ is real, the potential A_μ can also be taken as real. Substitution of (4.1) into (2.9) leads to an identity, and therefore (2.9) can now be disregarded, while substitution of (4.1) into (2.8) gives the field equation for the electromagnetic potential

$$\partial_\mu \partial_\nu A_\nu - \partial^2 A_\mu = \frac{1}{c} j_\mu. \tag{4.2}$$

According to (4.1), A_μ is not uniquely determined by the electromagnetic field tensor $F_{\mu\nu}$. Indeed, $F_{\mu\nu}$ remains unchanged under the transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{4.3}$$

where Λ is an arbitrary space-time function. The above transformation of the electromagnetic potential is called the gauge transformation.

The gauge transformation can be used to simplify the field equation (4.2). For, let $\bar{A}_\mu$ represent a possible value of the electromagnetic potential for a given value of $F_{\mu\nu}$, and let us consider a function Λ such that

$$\partial^2 \Lambda = -\partial_\mu \bar{A}_\mu.$$

Since it is known that the above equation can be solved for Λ, it is always possible to find the required function Λ. If a gauge transformation is now carried out such that the new electromagnetic potential is given by

$$A_\mu = \bar{A}_\mu + \partial_\mu \Lambda,$$

we find that

$$\partial_\mu A_\mu = 0. \tag{4.4}$$

Thus, the electromagnetic potential for a given electromagnetic field can always be chosen in such a way that (4.4) is satisfied. The relation (4.4) is called the supplementary condition, and with the use of this condition (4.2) reduces to the inhomogeneous wave equation

$$\partial^2 A_\mu = -\frac{1}{c} j_\mu. \tag{4.5}$$

Even when the supplementary condition is used, the electromagnetic potential is not determined by the electromagnetic field. For, both (4.1) and (4.4) remain unchanged under the transformation (4.3) provided that Λ

satisfies the wave equation

$$\partial^2 \Lambda = 0. \tag{4.6}$$

The transformation of the electromagnetic potential given by (4.3) and (4.6) represents a restricted gauge transformation.

5 Radiation Field

According to the Maxwell equations, the electromagnetic field depends on presence of charges in the field. But, even in the absence of charges the electromagnetic field does not in general vanish, because then (4.5) only reduces to the homogeneous wave equation

$$\partial^2 A_\mu = 0. \tag{5.1}$$

A general solution of (5.1) can be obtained by a superposition of plane waves of the form

$$A_\mu = a_\mu [e^{i(\mathbf{k}\cdot\mathbf{x} - k_0 ct + \delta)} + e^{-i(\mathbf{k}\cdot\mathbf{x} - k_0 ct + \delta)}] \tag{5.2}$$

with

$$|\mathbf{k}| = k_0, \tag{5.3}$$

which represents a plane wave traveling with the velocity c in the direction of the vector $\mathbf{k}$. Since (5.2) must satisfy the supplementary condition (4.4), it follows that

$$k_i a_i - k_0 a_0 = 0. \tag{5.4}$$

We regard $\mathbf{A}$ as consisting of a longitudinal component along $\mathbf{k}$ and two transverse components perpendicular to $\mathbf{k}$, and we call A_0 the temporal component of A_μ. In order to examine the properties of these various components, it is convenient to choose the space axes in such a way that the x_3 axis is along $\mathbf{k}$. Then, A_1 and A_2 are the transverse components of A_μ, while A_3 and A_0 are the longitudinal and temporal components, respectively. With such a choice of the axes,

$$k_1 = k_2 = 0, \qquad k_3 = |\mathbf{k}| = k_0, \tag{5.5}$$

so that (5.2) becomes

$$A_\mu = a_\mu [e^{ik_0(x_3 - ct) + i\delta} + e^{-ik_0(x_3 - ct) - i\delta}], \tag{5.6}$$

while the supplementary condition (5.4) takes the form

$$a_3 - a_0 = 0. \tag{5.7}$$

Using (4.1), we can find the electromagnetic field $F_{\mu\nu}$ due to the potential

(5.6), and then, with the help of (2.7), we can write its various components as

$$\begin{aligned} H_1 &= -k_0 a_2 f, \quad & H_2 &= k_0 a_1 f, \quad & H_3 &= 0, \\ E_1 &= k_0 a_1 f, \quad & E_2 &= k_0 a_2 f, \quad & E_3 &= k_0(a_3 - a_0) f = 0, \end{aligned} \tag{5.8}$$

with

$$f = i[e^{ik_0(x_3 - ct) + i\delta} - e^{-ik_0(x_3 - ct) - i\delta}], \tag{5.9}$$

where the component E_3 in (5.8) vanishes because of the supplementary condition (5.7). Thus, even in the absence of charges the electromagnetic field can contain plane electromagnetic waves, which possess energy as can be seen by substituting (5.8) into (3.10). An electromagnetic field that does not contain any charges is called a pure electromagnetic field or a radiation field.

According to (5.8),

$$E_1 H_1 + E_2 H_2 + E_3 H_3 = 0, \qquad E_3 = H_3 = 0, \tag{5.10}$$

which shows that the electric and magnetic fields due to a plane electromagnetic wave are perpendicular to each other as well as to the direction of propagation of the wave. The plane perpendicular to the electric field **E** of such an electromagnetic wave is taken as its plane of polarization. The relations (5.8) further show that in a radiation field the electric and magnetic field strengths depend only on the transverse components of the electromagnetic potential. Therefore, the longitudinal and temporal components of the electromagnetic potential do not give rise to any physical effect in a radiation field. In fact, in a radiation field these components can also be eliminated by subjecting (5.6) to the gauge transformation (4.3) with

$$\Lambda = \frac{ia_0}{k_0}[e^{ik_0(x_3 - ct) + i\delta} - e^{-ik_0(x_3 - ct) - i\delta}], \tag{5.11}$$

and then making use of (5.7). It must, however, be remembered that the longitudinal and temporal components of the electromagnetic potential cannot be ignored in the presence of charges.

6 Variational Principle for Classical Fields

In the preceding sections, a brief account of the classical electromagnetic field has been given. We shall now describe some general principles of the classical theory of fields.

In classical physics a field is described by one or more space-time functions satisfying certain partial differential equations, which are called the field equations. It is generally believed that the fields of physical interest can be described only by those field equations that can be obtained by means of Hamilton's variational principle in the following way:

Let L be a Lorentz-invariant function of any number of linearly independent field variables $u^{(1)}, u^{(2)}, \cdots, u^{(n)}$ and their first space and time derivatives, so that

$$L = L(u^{(r)}, \partial_\mu u^{(r)}), \tag{6.1}$$

where r takes the values $1, 2, \cdots, n$. Further, let us denote the integral of L over an arbitrary volume V and an arbitrary time interval t_1 to t_2 by

$$I = \int_{t_1}^{t_2} dt \int_V d\mathbf{x}\, L, \tag{6.2}$$

and assume that the above integral is stationary for any arbitrary infinitesimal variations of the $u^{(r)}$ provided that these variations vanish at the boundary of the domain of integration, i.e.,

$$\delta I = \int_{t_1}^{t_2} dt \int_V d\mathbf{x}\, \delta L = 0 \tag{6.3}$$

for the arbitrary infinitesimal variations

$$u^{(r)} \to u^{(r)} + \delta u^{(r)} \tag{6.4}$$

with

$$\begin{aligned} \delta u^{(r)} &= 0 \qquad \text{at the surface of } V, \\ &= 0 \qquad \text{for } t = t_1 \text{ and } t = t_2. \end{aligned} \tag{6.5}$$

According to the above variational principle, we obtain from (6.3), with the use of (6.1),

$$\sum_{r=1}^{n} \int_{t_1}^{t_2} dt \int_V d\mathbf{x} \left\{ \frac{\partial L}{\partial u^{(r)}} \delta u^{(r)} + \frac{\partial L}{\partial(\partial_\mu u^{(r)})} \delta(\partial_\mu u^{(r)}) \right\} = 0, \tag{6.6}$$

But,

$$\begin{aligned} \frac{\partial L}{\partial(\partial_\mu u^{(r)})} \delta(\partial_\mu u^{(r)}) &= \frac{\partial L}{\partial(\partial_\mu u^{(r)})} \partial_\mu(\delta u^{(r)}) \\ &= -\partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu u^{(r)})} \right] \delta u^{(r)} + \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu u^{(r)})} \delta u^{(r)} \right], \end{aligned} \tag{6.7}$$

so that (6.6) can be expressed as

$$\begin{aligned} \sum_r \int_{t_1}^{t_2} dt \int_V d\mathbf{x} \Bigg\{ & \frac{\partial L}{\partial u^{(r)}} \delta u^{(r)} - \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu u^{(r)})} \right] \delta u^{(r)} \\ & + \partial_i \left[\frac{\partial L}{\partial(\partial_i u^{(r)})} \delta u^{(r)} \right] + \partial_0 \left[\frac{\partial L}{\partial(\partial_0 u^{(r)})} \delta u^{(r)} \right] \Bigg\} = 0. \end{aligned} \tag{6.8}$$