美国数学会经典影印系列
AMS
AMERICAN MATHEMATICAL SOCIETY

Invariant Measures

不变测度

John von Neumann

高等教育出版社

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图字：01-2016-2498 号

不变测度
Bubian Cedu

图书在版编目 (CIP) 数据

不变测度 = Invariant Measures : 英文 / (美) 约翰 · 冯 · 诺伊曼 (John von Neumann) 著 . — 影印本 . — 北京 : 高等教育出版社 , 2017.4
ISBN 978-7-04-046997-4
Ⅰ. ①不… Ⅱ. ①约… Ⅲ. ①不变测度—英文
Ⅳ. ①O174.12
中国版本图书馆 CIP 数据核字 (2016) 第 326754 号

策划编辑 李 鹏　　责任编辑 李 鹏
封面设计 张申申　　责任印制 赵义民

出版发行 高等教育出版社
社址 北京市西城区德外大街 4 号
邮政编码 100120
购书热线 010-58581118
咨询电话 400-810-0598
网址 http://www.hep.edu.cn
http://www.hep.com.cn
网上订购 http://www.hepmall.com.cn
http://www.hepmall.com
http://www.hepmall.cn
印刷 北京中科印刷有限公司

开本 787mm × 1092 mm 1/16
印张 9.75
字数 240 千字
版次 2017 年 4 月第 1 版
印次 2017 年 4 月第 1 次印刷
定价 67.00 元

[物 料 号 46997-00]

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高 等 教 育 出 版 社

2016 年 12 月

Preface

In 1940–1941 von Neumann lectured on invariant measures at the Institute for Advanced Study. This book is essentially a written version of what he said.

The lectures began with general measure theory and went on to Haar measure and some of its generalizations. Shizuo Kakutani was at the Institute that year, and he and von Neumann had many conversations on the subject. The conversations revealed facts and produced proofs—quite a bit of the content of the course, especially toward the end, was discovered just a week or two or three before it appeared on the blackboard. The original version of these notes was prepared by Paul Halmos, von Neumann's assistant that year. Von Neumann read the handwritten version before it went to the typist, and sometimes scribbled comments on the margins. On Chapter VI, the last one, he did more than scribble—he himself wrote most of it.

The notes were typed. Two or three copies were kept in the Institute—von Neumann had one and the Institute library had another. Since then a few photocopies have been made, but until now the notes have never been published in any proper sense of the word.

Publisher's Note

This publication was made from a copy of a manuscript titled, *Invariant Measures* by John von Neumann, Notes by Paul R. Halmos. The copy, made by Roy L. Adler, was a xerographic copy of a xerographic copy of an ozalid copy (a copying process predating xerography) of a mimeograph copy of the first five chapters and a carbon copy of the sixth. The mimeograph copy and the carbon copy were supplied by Shizuo Kakutani. As a result of copying copies of copies, a certain amount of degradation had taken place, making some of the math difficult to read, and there are some errors in the original manuscript. To have a chance at catching them all one would have had to go over the galleys with an author's dedication. No one volunteered for that kind of labor. However, we thank Roy Adler, Bruce Kitchens, Karl Petersen, and Benjamin Weiss for their substantial efforts in proofreading.

We believe most introduced typos have been caught. Those found and believed to be errors by the author have been corrected.

Another problem with the original manuscript was the use of set theory notation no longer in fashion as well as notational and typographical inconsistencies. For example, set inclusion sometimes appears as $\subseteq$ and other times as $\leq$. The manuscript was prepared in the days before word processing and TEX, and symbols were inserted by hand. Different hands were at work in inscribing them. The handwriting in Chapters IV, V, and VI is not the same as that in the first three chapters. It appears to be Halmos's handwriting first and von Neumann's later. In addition, Chapter VI is quite different in its style, notation, and numbering scheme, and almost independent from the previous chapters. Since the author could not give us his wishes regarding changes, we have kept the old fashioned usage and inconsistencies to preserve a sense of history, and in the belief that no confusion will result. We have included examples of six pages, one from each chapter, from the fourth-generation manuscript used to create this volume: page 7 from Chapter I; page 39 from Chapter II; page 56 from Chapter III; page 81 from Chapter IV; page 136 from Chapter V; and page 151 from Chapter VI. The numbers inscribed on the last chapter were not put there by von Neumann, however. Originally, the carbons were unnumbered so Adler had to number them to prevent disarray.

§3. Measurability

(3.1.1) A set M is measurable if for every set K we have

$$\nu(K) = \nu(KM) + \nu(K\tilde{M}).$$

(3.1.2) In addition to measurable sets it will also be convenient to consider measurable partitions. A partition is a finite or countable sequence of pairwise disjoint sets whose sum is S. If $\mathfrak{A}$ is the partition $(A_1, A_2, \ldots)$ and $\mathfrak{B} = (B_1, B_2, \ldots)$, we write $\mathfrak{A} \leq \mathfrak{B}$ if every A is a subset of some B. Under this partial ordering the set of all partitions is a lattice: i.e. to every pair $\mathfrak{A}$, $\mathfrak{B}$ of partitions there corresponds a unique partition $\mathfrak{C}$ (called the product of $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{C} = \mathfrak{A} \cdot \mathfrak{B}$) with the properties that $\mathfrak{C} \leq \mathfrak{A}$, $\mathfrak{C} \leq \mathfrak{B}$, and $\mathfrak{C}' \leq \mathfrak{A}$, $\mathfrak{C}' \leq \mathfrak{B}$ implies $\mathfrak{C}' \leq \mathfrak{C}$. $\mathfrak{C}$ is the partition whose sets are $A_i B_j$, $i, j = 1, 2, \ldots$. A partition $\mathfrak{A} = (A_1, A_2, \ldots)$ is measurable if for every set K we have

$$\nu(K) = \nu(KA_1) + \nu(KA_2) + \ldots .$$

We observe that M is a measurable set if and only if the partition $(M, \tilde{M})$ is a measurable partition.

(3.2) If M is such that $\nu(O) \geq \nu(OM) + \nu(O\tilde{M})$ for every open set O, then M is measurable.

Proof. Let K be an arbitrary set, and O an open set, $O \supseteq K$. Then

$$\nu(O) \geq \nu(OM) + \nu(O\tilde{M}) \geq \nu(KM) + \nu(K\tilde{M}).$$

Since $\nu(O) = \mu(O)$, we have

$$\mu(O) \geq \nu(KM) + \nu(K\tilde{M})$$

for all $O \supseteq K$, so that

$$\nu(K) = \inf \mu(O) \geq \nu(KM) + \nu(K\tilde{M}).$$

The opposite inequality follows from (2.8.1).

FIGURE 1. Chapter I, page 7

<u>Proof.</u> Let $\mathfrak{B}$ be an arbitrary family of closed sets $F \subseteq S$, such that for $F_1, \ldots, F_n \in \mathfrak{B}$ always $F_1 \cdot \ldots \cdot F_n \neq \Theta$. By adding all these sets $F_1 \cdot \ldots \cdot F_n$ to $\mathfrak{B}$ we see that there is no loss of generality in assuming that $F, G \in \mathfrak{B}$ imply $F \cdot G \in \mathfrak{B}$. And still $\Theta \notin \mathfrak{B}$.

For each $F \in \mathfrak{B}$ select an element x_F^o of F.

Consider a family $\mathfrak{B}' \subseteq \mathfrak{B}$ with this property:

(10.7.1.3) There exists an $F_o \in \mathfrak{B}$ such that $F \in \mathfrak{B}'$ implies $x_F^o \notin F_o$.

The set $\mathfrak{J}$ of all such $\mathfrak{B}'$ is an ideal of subsets of $\mathfrak{B}$: That $\mathfrak{B}' \in \mathfrak{J}$ and $\mathfrak{B}'' \subseteq \mathfrak{B}'$ imply $\mathfrak{B}'' \in \mathfrak{J}$ is clear. And if $\mathfrak{B}', \mathfrak{B}'' \in \mathfrak{J}$ then $\mathfrak{B}' + \mathfrak{B}'' \in \mathfrak{J}$ because if (10.7.1.3) holds for $\mathfrak{B}'$ with F_o', and for $\mathfrak{B}''$ with F_o'', then it holds for $\mathfrak{B}' + \mathfrak{B}''$ with $F_o' \cdot F_o''$. Furthermore $x_F^o \in F$ excludes that (10.7.1.3) be true for $\mathfrak{B}$ with any F_o, hence $\mathfrak{B} \notin \mathfrak{J}$, i.e. $\mathfrak{J} \neq \mathbf{I}$.

So we may apply our hypothesis to $I = \mathfrak{B}$ and this $\mathfrak{J}$ and obtain a function $\varphi(\xi) = \varphi(x_F | F \in \mathfrak{B})$ which fulfills (10.7.1.1), (10.7.1.2). Since $x_F^o \in F \subseteq C$, we can form $\varphi(x_F^o | F \in \mathfrak{B})$.

Consider an $F_o \in \mathfrak{B}$. Let $\mathfrak{B}'$ be the set of all $F \in \mathfrak{B}$ with $x_F^o \notin F_o$. Then $\mathfrak{B}' \in \mathfrak{J}$. Choose a $G_o \notin \mathfrak{B}'$ and form $\xi' = (x_F' | F \in \mathfrak{B})$ with

$$x_F' = x_F^o \text{ for } F \notin \mathfrak{B}',$$
$$x_F' = x_{G_o}^o \text{ for } F \in \mathfrak{B}'.$$

Then every x_F' is an x_G^o with $G \notin \mathfrak{B}'$, i.e. with $x_G^o \in F_o$, so

FIGURE 2. Chapter II, page 39

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63 77
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Chapter III

Haar measure

§13. Remarks on measures

We return to the considerations and notations of §2. We assume that S is a Hausdorff space, $\lambda(C)$ is a set function defined for all compact sets C, and $\nu(M)$ is the measure generated by $\lambda(C)$ (cf. 4.2). For convenience of reference we give below a set of properties of the space S, and the functions $\lambda(C)$ and $\nu(M)$, and then we establish certain implication relations among them.

(13.1.1) $0 \leqq \lambda(C) \leqq \infty$.

(13.1.2) $\lambda(C+D) \leqq \lambda(C) + \lambda(D)$.

(13.1.3) If $CD = \Theta$, $\lambda(C+D) = \lambda(C) + \lambda(D)$.

(13.1.4) If $C \subseteqq D$, $\lambda(C) \leqq \lambda(D)$.

(13.1.5) If $C^i \neq \Theta$, $\lambda(C) > 0$.

(13.1.6) $\lambda(C) < \infty$.

(13.1.7) $x \to \varphi(x)$ is a homeomorphism of S into itself for which $\lambda(\varphi(C)) = \lambda(C)$ for all C.

(13.2) S is locally compact.

(13.2.1) $0 \leqq \nu(M) \leqq \infty$.

(13.3.2) $\nu(\sum_{i=1}^{\infty} M_i) \leqq \sum_{i=1}^{\infty} \nu(M_i)$.

(13.3.3) If $\{M_j\}$ is a sequence of measurable sets (cf. (13.1.1)) such that for $k \neq j$, $M_k \cdot M_j = \Theta$, then $\nu(\sum_{j=1}^{\infty} M_j) = \sum_{j=1}^{\infty} \nu(M_j)$.

(13.3.4) If $M^i \neq \Theta$, $\nu(M_i) > 0$.

(13.3.5) If $\overline{M}$ is compact, $\nu(M) < \infty$.

(13.3.6) $x \to \varphi(x)$ is a homeomorphism of S into itself for which $\nu(\varphi(M)) = \nu(M)$ for all M.

FIGURE 3. Chapter III, page 56

(17.7) <u>If $\mathfrak{A}$ is a class of type $\mathcal{D}$ then $\mathcal{F}(\mathfrak{A}) = \mathcal{D}'(\mathfrak{A})$.</u>

<u>Proof.</u> It is clear that $\mathcal{D} = \mathcal{D}'(\mathfrak{A}) \subseteq \mathcal{F} = \mathcal{F}(\mathfrak{A})$; we shall prove that $\mathcal{D} = \mathcal{F}$ by showing that $\mathcal{D}$ is a field.

(17.7.1) It is clear from the definition of $\mathcal{D}(\mathfrak{A})$ that $D_i \in \mathcal{D}$ for $i = 1,\ldots,n$, $D_iD_j = \Theta$ for $i \neq j$, implies $D_1 + \ldots + D_n \in \mathcal{D}$

(17.7.2) If $A,B \in \mathfrak{A}$ then $A\tilde{B} \in \mathcal{D}$

(17.7.3) If $A \in \mathcal{D}$ and $B \in \mathfrak{A}$ then $A\tilde{B} \in \mathcal{D}$ For by hypothesis we may write A as a disjoint sum of sets of $\mathfrak{A}$ $A = A_1 + A_2 + \ldots + A_n$, so that

$$A\tilde{B} = (A_1 + \ldots + A_n)\tilde{B} = A_1\tilde{B} + \ldots + A_n\tilde{B}.$$

By (17.7.2), $A_i\tilde{B} \in \mathcal{D}$ and it follows from (17.7.1) that $A\tilde{B} \in \mathcal{D}$

(17.7.4) If $A,B \in \mathcal{D}$ then $A\tilde{B} \in \mathcal{D}$. For if $B = B_1 + \ldots + B_m$, where the B_i are pairwise disjoint sets of $\mathfrak{A}$, then

$$A\tilde{B} = A(\widetilde{B_1 + \ldots + B_m}) = A\tilde{B}_1\tilde{B}_2 \ldots \tilde{B}_m,$$

and the desired result follows by repeated application of (17.7.3).

(17.7.5) If $A,B \in \mathcal{D}$ $A+B \in \mathcal{D}$. For we have $A+B = A\tilde{B}+B$. The last sum has disjoint addends which (by (17.7.4)) belong to $\mathcal{D}$ hence (by (17.7.1)) it belongs to $\mathcal{D}$

Together the statements (17.7.4) and (17.7.5) merely assert that is a field, as was to be proved.

(17.8) <u>If $\mathcal{F}$ is a field, then $\mathcal{B}(\mathcal{F}) = \mathfrak{M}(\mathcal{F})$</u>

<u>Proof.</u> The structure of this proof is similar to the one given above. We observe that $\mathfrak{M} = \mathfrak{M}(\mathcal{F}) \subseteq \mathcal{B} = \mathcal{B}(\mathcal{F})$ and we shall complete the proof by showing that $\mathfrak{M}$ is a Borel field. We remark that it is sufficient to prove that $\mathfrak{M}$ is a field. For if $\mathfrak{M}$ is a field and $A_i \in \mathfrak{M}$, $i = 1,2,\ldots$, then $A_i' = A_1 + \ldots + A_i \in \mathfrak{M}$, whence (since $\mathfrak{M}$

FIGURE 4. Chapter IV, page 81

(25.1.1) Given two $f, g \in \mathfrak{H}$ and an $\epsilon > 0$, denote by $N_1^c(f, g; \epsilon)$ the set of all $b \in \mathcal{J}$ with

$$|(U_b f, g)| \geq \epsilon.$$

Then A has a compact closure if and only if some $N_1^c(f, g; \epsilon) \geqq A$.

(25.1.2) The same is true if we restrict ourselves to the $N_1^c(f, f; \epsilon)$ with $f = g$. We can also assume that $|f| = 1$.

(25.1.3) For $|f| = 1$ the above $N_1^c(f, f; \epsilon)$ is the set of all $b \in \mathcal{J}$ with

(25.1.3.1) $$|(U_b f, f)| \geq \epsilon$$ We can replace it by the set $N_2^c(f, \epsilon)$ of all $b \in \mathcal{J}$ with (25.1.3.2) $\mathcal{B}(U_b f, f) \geq \epsilon$.

(25.1.4) Given two Borel sets $M, N \in \mathfrak{L}_{\mathfrak{H}}$ and an $\epsilon > 0$, denote by $N_3^c(M, N; \epsilon)$ the set of all $b \in \mathcal{J}$ with

$$\nu(bM \cdot N) \geq \epsilon$$

Then A has a compact closure if and only if some $N_3^c(M, N; \epsilon) \geqq A$.

(25.1.5) The same is true if we restrict ourselves to the $N_3^c(M, M; \epsilon)$ with $M = N$.

Proof: We must prove two things:

(α) Each one of the above sets N^c has a compact closure.

(β) If C is compact, then there exists a set $N^c \geqq C$, for each one of the above described categories of sets N

Proof of (α): In this case it suffices to prove (25.1.1), the others are special cases of this. Indeed: (25.1.2) is a special case of (25.1.1). In (25.1.3), (25.1.3.1) is an obvious restatement of the definition of $N_1^c(f, f; \epsilon)$. The $N_2^c(f; \epsilon)$ of (25.1.3.2) may be used since (25.1.3.2) implies (25.1.3.1) so that $N_1^c(f, f; \epsilon) \geqq N_2^c(f; \epsilon)$. (25.1.4) is a special case of (25.1.1), with $f = \chi_M$, $g = \chi_N$. (25.1.5) is a special case of (25.1.4).

FIGURE 5. Chapter V, page 136

(30) (28) implies $j = k''$ for $k = 1, 2, \cdots, \tilde{p}+1$. Thus $\alpha) - \gamma)$ are satisfied (by (26), (29), (30)) with $p = \tilde{p}+1$. This contradicts our original assumption.

Thus all alternatives and Rules are exhausted, and the proof is completed.

3. Notations (Topology and Group Theory)

G : Topological group.

$x\,y$: Composition rule (in G).

x^{-1} : Reciprocal (in G).

1 : Unit (in G).

M, N : Arbitrary subset of G.

O, P, Q : Open subset of G.

C, D, E : Compact subsets of G.

$\bar{M}$: Closure of M (in G).

M^i : Interior of M (in G).

xM : Set $(xu \mid u \in M)$.

Mx : Set $(ux \mid u \in M)$.

M^{-1} : Set $(u^{-1} \mid u \in M)$.

Hypotheses:

1) x^{-1} is a continuous (1-variable) function of x (in all G).

2) xy is a continuous (2-variable) function of x, y (in all G).

3) G is locally compact; i.e., there exists a C with $1 \in C^i$.

4. Equidistribution

Let a C be given which will remain fixed throughout all our discussions.

We define:

FIGURE 6. Chapter VI, p. 151

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