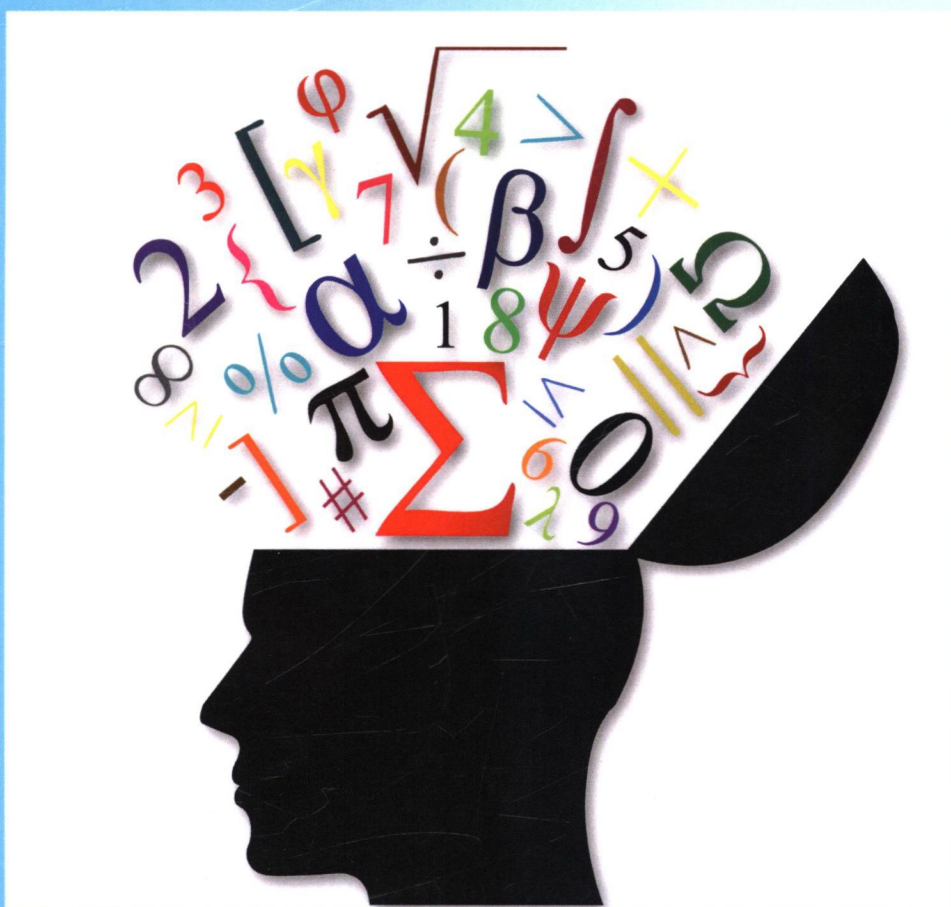


TEXTBOOKS IN MATHEMATICS

ESSENTIALS OF MATHEMATICAL THINKING



Steven G. Krantz



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ESSENTIALS OF MATHEMATICAL THINKING


Essentials of Mathematical Thinking addresses the growing need to better comprehend mathematics today. Increasingly, our world is driven by mathematics in all aspects of life. The book is an excellent introduction to the world of mathematics for students not majoring in mathematical studies.

The author has written this book in an enticing, rich manner with helpful graphics that will engage students and introduce new paradigms of thought. Careful readers will develop critical thinking skills which will help them compete in today's world.

The book explains:

- What goes behind a Google search algorithm
- How to calculate the odds in a lottery
- The value of Big Data
- How the nefarious Ponzi scheme operates

Instructors will treasure the book for its ability to make the field of mathematics more accessible and alluring with relevant topics and helpful graphics. The author encourages readers to see the beauty of mathematics and how it relates to their lives in meaningful ways.

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To the memory of Ed Nelson.

Preface

It is becoming increasingly clear that mathematical thinking is essential to understanding the world around us. Police tell us that most crimes these days are committed with electronic devices (usually computers). Many if not most medical procedures are now robotically assisted. The carburation system of a modern automobile is all electronic. Geometric visualization is one of the most important aspects of modern technology.

And it is plain that the thinking behind all of the technology described in the last paragraph is mathematical. The **Google** search engine is based on a mathematical algorithm. Not just logic, but analysis and combinatorics and geometry and many other ideas from the mathematical sciences are critical parts of our world.

A college education consists of learning different modes of discourse. Every modern student should thus have some exposure to mathematical discourse. Whether or not the student has any intention of working in the mathematical sciences, he/she will need these modes of thought to survive in the modern marketplace.

And learning mathematical thought does not have to be dry, desultory, and boring. It can be fun and exciting and challenging (in a productive fashion). It can relate to things that are meaningful to students—such as the paradigm behind a Google search, or the means by which a music CD is encoded, or the RSA encryption algorithm. In a course on mathematical thinking the student will obviously learn some mathematical skills, but he/she will also gain *critical thinking skills*. These are valuable life tools.

This book may be thought of as a book for nonmathematicians taking

a mathematics course that is a breadth requirement for an undergraduate degree. Thus it has minimal mathematical prerequisites. It has very few equations. It has lots of pictures and lots of explanation. It is driven by examples. But those examples will lead students down new paths, and acquaint them with new paradigms of thought. They will stimulate and provoke and encourage. At the end of this journey, the student should feel that he/she has a new set of tools for attacking a variety of problems and situations. There should be a profound feeling of satisfaction.

A very important part of learning mathematical thinking is learning to write mathematics. The process of internalizing an idea and the process of figuring out how to write it down are closely linked. We exploit this observation by supplying on-the-fly exercises that force the student to *write out* an answer. Sometimes, when it is appropriate and not burdensome, we ask the student to provide a demonstration or a reason for a mathematical idea. We also ask the student to draw pictures.

The exercises in this book are of several types: **(a)** there are drill exercises, just as a means of getting students started in a topic, **(b)** there are more challenging problems that will require the student to really digest the key ideas, and **(c)** each section ends with an open-ended problem that will encourage students to talk to each other and collaborate.

The material in the first nine chapters of the book should be accessible to a broad audience with minimal background in mathematics. The last three chapters are more sophisticated. All readers will want to dip into these chapters, but only readers with some mathematical background will be able to work through the examples in any detail. In this way, we have been able to make the book more open-ended and appealing to a diverse group of readers with many needs and interests.

It is a pleasure to thank Lynn Apfel and Tim Davis for helpful remarks and suggestions for many different parts of this book. I thank the many fine reviewers that Taylor & Francis engaged to criticize my book. As always, my editor Robert Ross was supportive and proactive in the production of this book.

Steven G. Krantz
St. Louis, Missouri

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