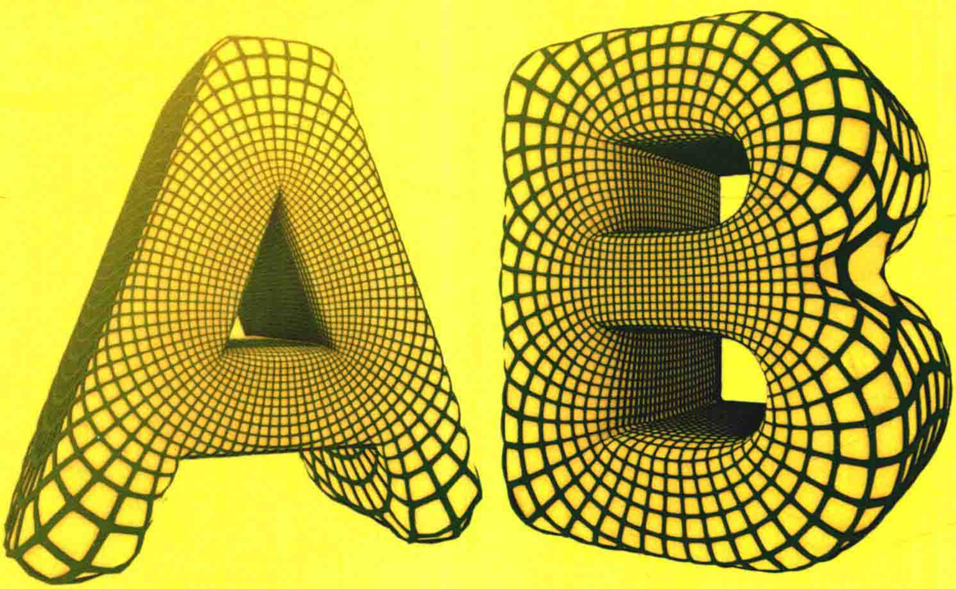
Alexander I. Bobenko *Editor*

Advances in Discrete Differential Geometry



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Preface

In this book we take a closer look at discrete models in differential geometry and dynamical systems. The curves used are polygonal, surfaces are made from triangles and quadrilaterals, and time runs discretely. Nevertheless, one can hardly see the difference to the corresponding smooth curves, surfaces, and classical dynamical systems with continuous time. This is the paradigm of structure-preserving discretizations. The common idea is to find and investigate discrete models that exhibit properties and structures characteristic of the corresponding smooth geometric objects and dynamical processes. These important and characteristic qualitative features should already be captured at the discrete level. The current interest and advances in this field are to a large extent stimulated by its relevance for computer graphics, mathematical physics, architectural geometry, etc.

The book focuses on differential geometry and dynamical systems, on smooth and discrete theories, and on pure mathematics and its practical applications. It demonstrates this interplay using a range of examples, which include discrete conformal mappings, discrete complex analysis, discrete curvatures and special surfaces, discrete integrable systems, special texture mappings in computer graphics, and freeform architecture. It was written by specialists from the DFG Collaborative Research Center "Discretization in Geometry and Dynamics". The work involved in this book and other selected research projects pursued by the Center was recently documented in the film "The Discrete Charm of Geometry" by Ekaterina Eremenko.

Lastly, the book features a wealth of illustrations, revealing that this new branch of mathematics is both (literally) beautiful and useful. In particular the cover illustration shows the discretely conformally parametrized surfaces of the inflated letters A and B from the recent educational animated film "conform!" by Alexander Bobenko and Charles Gunn.

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Contents

Discrete Conformal Maps: Boundary Value Problems, Circle Domains, Fuchsian and Schottky Uniformization

Alexander I. Bobenko, Stefan Sechelmann and Boris Springborn

Abstract We discuss several extensions and applications of the theory of discretely conformally equivalent triangle meshes (two meshes are considered conformally equivalent if corresponding edge lengths are related by scale factors attached to the vertices). We extend the fundamental definitions and variational principles from triangulations to polyhedral surfaces with cyclic faces. The case of quadrilateral meshes is equivalent to the cross ratio system, which provides a link to the theory of integrable systems. The extension to cyclic polygons also brings discrete conformal maps to circle domains within the scope of the theory. We provide results of numerical experiments suggesting that discrete conformal maps converge to smooth conformal maps, with convergence rates depending on the mesh quality. We consider the Fuchsian uniformization of Riemann surfaces represented in different forms: as immersed surfaces in $\mathbb{R}^3$, as hyperelliptic curves, and as $\mathbb{CP}^1$ modulo a classical Schottky group, i.e., we convert Schottky to Fuchsian uniformization. Extended examples also demonstrate a geometric characterization of hyperelliptic surfaces due to Schmutz Schaller.

1 Introduction

Not one, but several sensible definitions of discrete holomorphic functions and discrete conformal maps are known today. The oldest approach, which goes back to the early finite element literature, is to discretize the Cauchy–Riemann equa-

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tions [10–14, 27]. This leads to linear theories of discrete complex analysis, which have recently returned to the focus of attention in connection with conformal models of statistical physics [8, 9, 22, 23, 29, 40–42], see also [4].

The history of nonlinear theories of discrete conformal maps goes back to Thurston, who introduced patterns of circles as elementary geometric way to visualize hyperbolic polyhedra [45, Chapter 13]. His conjecture that circle packings could be used to approximate Riemann mappings was proved by Rodin and Sullivan [35]. This initiated a period of intensive research on circle packings and circle patterns, which lead to a full-fledged theory of discrete analytic functions and discrete conformal maps [44].

A related but different nonlinear theory of discrete conformal maps is based on a straightforward definition of discrete conformal equivalence for triangulated surfaces: Two triangulations are discretely conformally equivalent if the edge lengths are related by scale factors assigned to the vertices. This also leads to a surprisingly rich theory [5, 17, 18, 28]. In this article, we investigate different aspects of this theory (Fig. 1).

We extend the notion of discrete conformal equivalence from triangulated surfaces to polyhedral surfaces with faces that are inscribed in circles. The basic definitions and their immediate consequences are discussed in Sect. 2.

In Sect. 3, we generalize a variational principle for discretely conformally equivalent triangulations [5] to the polyhedral setting. This variational principle is the main tool for all our numerical calculations. It is also the basis for our uniqueness proof for discrete conformal mapping problems (Theorem 3.9).

Section 4 is concerned with the special case of quadrilateral meshes. We discuss the emergence of orthogonal circle patterns, a peculiar necessary condition for the existence of solutions for boundary angle problems, and we extend the method of constructing discrete Riemann maps from triangulations to quadrangulations.

In Sect. 5, we briefly discuss discrete conformal maps from multiply connected domains to circle domains, and special cases in which we can map to slit domains.

Section 6 deals with conformal mappings onto the sphere. We generalize the method for triangulations to quadrangulations, and we explain how the spherical version of the variational principle can in some cases be used for numerical calculations although the corresponding functional is not convex.

Section 7 is concerned with the uniformization of tori, i.e., the representation of Riemann surfaces as a quotient space of the complex plane modulo a period lattice. We consider Riemann surfaces represented as immersed surfaces in $\mathbb{R}^3$, and as elliptic curves. We conduct numerical experiments to test the conjectured convergence of discrete conformal maps. We consider the difference between the true modulus of an elliptic curve (which can be calculated using hypergeometric functions) and the modulus determined by discrete uniformization, and we estimate the asymptotic dependence of this error on the number of vertices.

In Sect. 8, we consider the Fuchsian uniformization of Riemann surfaces represented in different forms. We consider immersed surfaces in $\mathbb{R}^3$ (and S^3), hyperelliptic curves, and Riemann surfaces represented as a quotient of $\hat{\mathbb{C}}$ modulo a classical Schottky group. That is, we convert from Schottky uniformization to Fuchsian uniformization. The section ends with two extended examples demonstrating, among

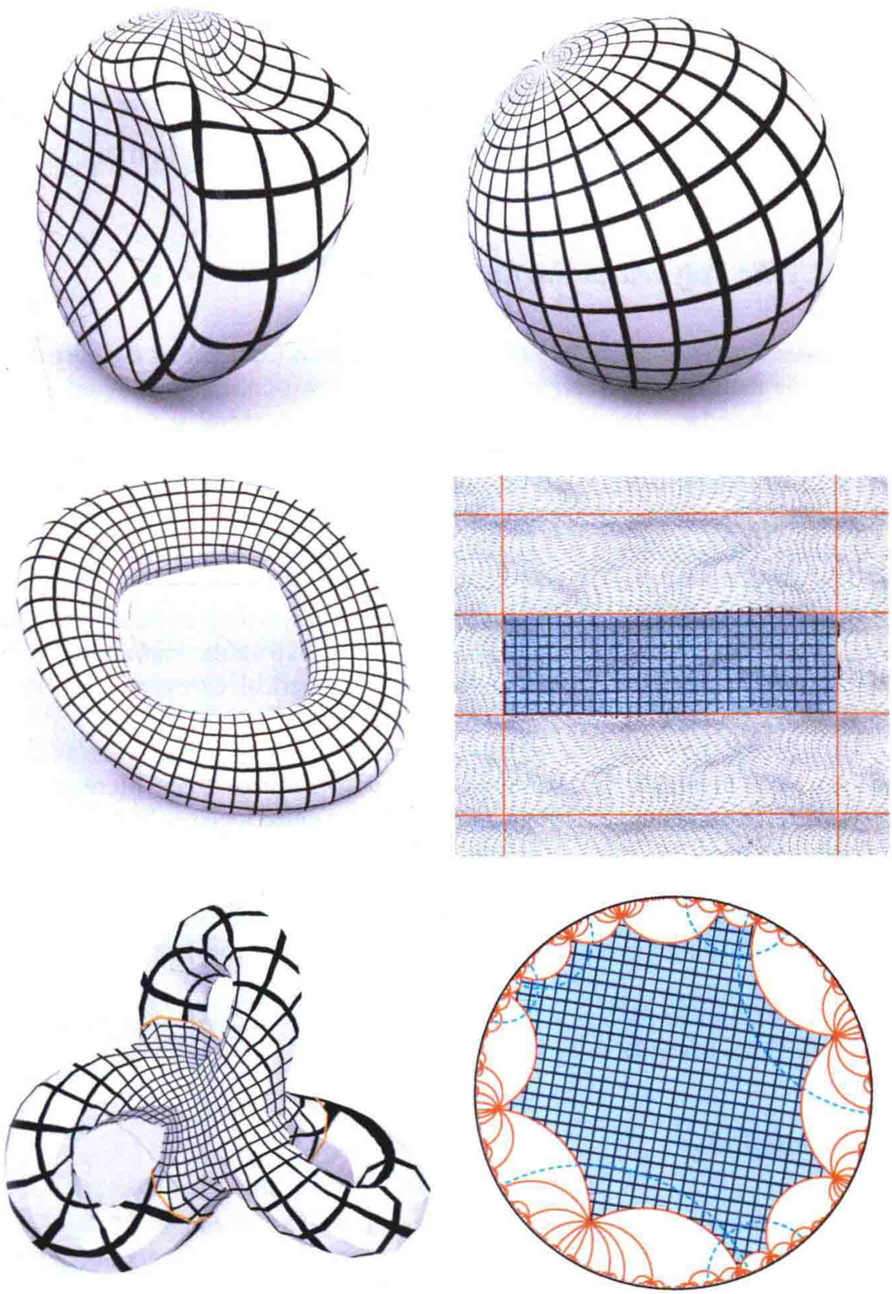

Fig. 1 Uniformization of compact Riemann surfaces. The uniformization of spheres is treated in Sect. 6. Tori are covered in Sect. 7, and Sect. 8 is concerned with surfaces of higher genus

other things, a remarkable geometric characterization of hyperelliptic surfaces due to Schmutz Schaller.

2 Discrete Conformal Equivalence of Cyclic Polyhedral Surfaces

2.1 Cyclic Polyhedral Surfaces

A *euclidean polyhedral surface* is a surface obtained from gluing euclidean polygons along their edges. (A *surface* is a connected two-dimensional manifold, possibly with boundary.) In other words, a euclidean polyhedral surface is a surface equipped with, first, an intrinsic metric that is flat except at isolated points where it has cone-like singularities, and, second, the structure of a CW complex with geodesic edges. The set of vertices contains all cone-like singularities. If the surface has a boundary, the boundary is polygonal and the set of vertices contains all corners of the boundary.

Hyperbolic polyhedral surfaces and *spherical polyhedral surfaces* are defined analogously. They are glued from polygons in the hyperbolic and elliptic planes, respectively. Their metric is locally hyperbolic or spherical, except at cone-like singularities.

We will only be concerned with polyhedral surfaces whose faces are all cyclic, i.e., inscribed in circles. We call them *cyclic polyhedral surfaces*. More precisely, we require the polygons to be cyclic before they are glued together. It is not required that the circumcircles persist after gluing; they may be disturbed by cone-like singularities. A polygon in the hyperbolic plane is considered cyclic if it is inscribed in a curve of constant curvature. This may be a circle (the locus of points at constant distance from its center), a horocycle, or a curve at constant distance from a geodesic.

A *triangulated surface*, or *triangulation* for short, is a polyhedral surface all of whose faces are triangles. All triangulations are cyclic.

2.2 Notation

We will denote the sets of vertices, edges, and faces of a CW complex Σ by V_Σ, E_Σ, and F_Σ, and we will often omit the subscript when there is no danger of confusion. For notational convenience, we require all CW complexes to be *strongly regular*. This means that we require that faces are not glued to themselves along edges or at vertices, that two faces are not glued together along more than one edge or one vertex, and that edges have distinct end-points and two edges have at most one endpoint in common. This allows us to label edges and faces by their vertices. We will write $ij \in E$ for the edge with vertices $i, j \in V$ and $ijkl \in F$ for the face with vertices $i, j, k, l \in V$. We will always list the vertices of a face in the correct cyclic order, so that for example the face $ijkl$ has edges ij, jk, kl, and li. The only reason for restricting our discussion to strongly regular CW complexes is to be able to use this simple notation. Everything we discuss applies also to general CW complexes.

2.3 Discrete Metrics

The *discrete metric* of a euclidean (or hyperbolic or spherical) cyclic polyhedral surface Σ is the function $\ell : E_\Sigma \to \mathbb{R}_{>0}$ that assigns to each edge $ij \in E_\Sigma$ its length ℓ_{ij}. It satisfies the polygon inequalities (one side is shorter than the sum of the others):

$$\left.\begin{aligned} -\ell_{i_1i_2} + \ell_{i_2i_3} + \ldots + \ell_{i_{n-1}i_n} > 0 \\ \ell_{i_1i_2} - \ell_{i_2i_3} + \ldots + \ell_{i_{n-1}i_n} > 0 \\ \vdots \\ \ell_{i_1i_2} + \ell_{i_2i_3} + \ldots - \ell_{i_{n-1}i_n} > 0 \end{aligned}\right\} \quad \text{for all } i_1i_2\ldots i_n \in F_\Sigma \qquad (1)$$

In the case of spherical polyhedral surfaces, we also require that

$$\ell_{i_1i_2} + \ell_{i_2i_3} + \ldots + \ell_{i_{n-1}i_n} < 2\pi. \qquad (2)$$

The polygon inequalities (1) are necessary and sufficient for the existence of a unique cyclic euclidean polygon and a unique cyclic hyperbolic polygon with the given edge lengths. Together with inequality (2) they are necessary and sufficient for the existence of a unique cyclic spherical polygon. For a new proof of these elementary geometric facts, see [24]. Thus, a discrete metric determines the geometry of a cyclic polyhedral surface:

Proposition and Definition 2.1 *If Σ is a surface with the structure of a CW complex and a function $\ell : E_\Sigma \to \mathbb{R}_{>0}$ satisfies the polygon inequalities* (1)*, then there is a unique euclidean cyclic polyhedral surface and also a unique hyperbolic cyclic polyhedral surface with CW complex Σ and discrete metric ℓ. If ℓ also satisfies the inequalities* (2)*, then there is a unique spherical cyclic polyhedral surface with CW complex Σ and discrete metric ℓ.*

We will denote the euclidean, hyperbolic, and spherical polyhedral surface with CW complex Σ and discrete metric ℓ by $(\Sigma, \ell)_{euc}$, $(\Sigma, \ell)_{hyp}$, and $(\Sigma, \ell)_{sph}$, respectively.

2.4 Discrete Conformal Equivalence

We extend the definition of discrete conformal equivalence from triangulations [5, 28] to cyclic polyhedral surfaces in a straightforward way (Definition 2.2). While some aspects of the theory carry over to the more general setting (e.g., Möbius invariance, Proposition 2.5), others do not, like the characterization of discretely conformally equivalent triangulations in terms of length cross-ratios (Sect. 2.5). We will discuss similar characterizations for polyhedral surfaces with 2-colorable vertices and the particular case of quadrilateral faces in Sects. 2.7 and 2.8.

We define discrete conformal equivalence only for polyhedral surfaces that are combinatorially equivalent (see Remark 2.4). Thus, we may assume that the surfaces share the same CW complex Σ equipped with different metrics ℓ, $\tilde{\ell}$.

Definition 2.2 *Discrete conformal equivalence* is an equivalence relation on the set of cyclic polyhedral surfaces defined as follows:

- Two *euclidean* cyclic polyhedral surfaces $(\Sigma, \ell)_{euc}$ and $(\Sigma, \tilde{\ell})_{euc}$ are *discretely conformally equivalent* if there exists a function $u : V_\Sigma \to \mathbb{R}$ such that

$$\tilde{\ell}_{ij} = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}. \tag{3}$$

- Two *hyperbolic* cyclic polyhedral surfaces $(\Sigma, \ell)_{hyp}$ and $(\Sigma, \tilde{\ell})_{hyp}$ are *discretely conformally equivalent* if there exists a function $u : V_\Sigma \to \mathbb{R}$ such that

$$\sinh\left(\frac{\tilde{\ell}_{ij}}{2}\right) = e^{\frac{1}{2}(u_i+u_j)} \sinh\left(\frac{\ell_{ij}}{2}\right). \tag{4}$$

- Two *spherical* cyclic polyhedral surfaces $(\Sigma, \ell)_{sph}$ and $(\Sigma, \tilde{\ell})_{sph}$ are *discretely conformally equivalent* if there exists a function $u : V_\Sigma \to \mathbb{R}$ such that

$$\sin\left(\frac{\tilde{\ell}_{ij}}{2}\right) = e^{\frac{1}{2}(u_i+u_j)} \sin\left(\frac{\ell_{ij}}{2}\right). \tag{5}$$

We will also consider mixed versions:

- A euclidean cyclic polyhedral surface $(\Sigma, \ell)_{euc}$ and a hyperbolic cyclic polyhedral surface $(\Sigma, \tilde{\ell})_{hyp}$ are discretely conformally equivalent if

$$\sinh\left(\frac{\tilde{\ell}_{ij}}{2}\right) = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}. \tag{6}$$

- A euclidean cyclic polyhedral surface $(\Sigma, \ell)_{euc}$ and a spherical cyclic polyhedral surface $(\Sigma, \tilde{\ell})_{sph}$ are discretely conformally equivalent if

$$\sin\left(\frac{\tilde{\ell}_{ij}}{2}\right) = e^{\frac{1}{2}(u_i+u_j)}\ell_{ij}. \tag{7}$$

- A hyperbolic cyclic polyhedral surface $(\Sigma, \ell)_{hyp}$ and a spherical cyclic polyhedral surface $(\Sigma, \tilde{\ell})_{sph}$ are discretely conformally equivalent if

$$\sin\left(\frac{\tilde{\ell}_{ij}}{2}\right) = e^{\frac{1}{2}(u_i+u_j)} \sinh\left(\frac{\ell_{ij}}{2}\right). \tag{8}$$

Remark 2.3 Note that relation (5) for spherical edge lengths is equivalent to relation (3) for the euclidean lengths of the chords in the ambient $\mathbb{R}^3$ of the sphere (see

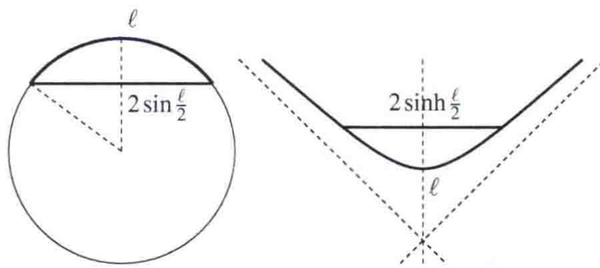

Fig. 2 Spherical and hyperbolic chords

Fig. 2, left). Likewise, relation (4) for hyperbolic edge lengths is equivalent to (3) for the euclidean lengths of the chords in the ambient $\mathbb{R}^{2,1}$ of the hyperboloid model of the hyperbolic plane (see Fig. 2, right).

Remark 2.4 For triangulations, the definition of discrete conformal equivalence has been extended to meshes that are not combinatorially equivalent [5, Definition 5.1.4] [17, 18]. It is not clear whether or how the following definitions for cyclic polyhedral surfaces can be extended to combinatorially inequivalent CW complexes.

The discrete conformal class of a cyclic polyhedral surface embedded in n-dimensional euclidean space is invariant under Möbius transformations of the ambient space:

Proposition 2.5 (Möbius invariance) *Suppose P and $\tilde{P}$ are two combinatorially equivalent euclidean cyclic polyhedral surfaces embedded in $\mathbb{R}^n$ (with straight edges and faces), and suppose there is a Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$ that maps the vertices of P to the corresponding vertices of $\tilde{P}$. Then P and $\tilde{P}$ are discretely conformally equivalent.*

Note that only vertices are related by the Möbius transformation, not edges and faces, which remain straight. The simple proof for the case of triangulations [5] carries over without change.

2.5 *Triangulations: Characterization by Length Cross-Ratios*

For euclidean triangulations, there is an alternative characterization of conformal equivalence in terms of length cross-ratios [5]. We review the basic facts in this section.

For two adjacent triangles $ijk \in F$ and $jil \in F$ (see Fig. 3), the *length cross-ratio* of the common interior edge $ij \in E$ is defined as

$$\operatorname{lcr}_{ij} = \frac{\ell_{il}\ell_{jk}}{\ell_{lj}\ell_{ki}}. \tag{9}$$

(If the two triangles are embedded in the complex plane, this is just the modulus of the complex cross-ratio of the four vertices.) This definition of length cross-ratios

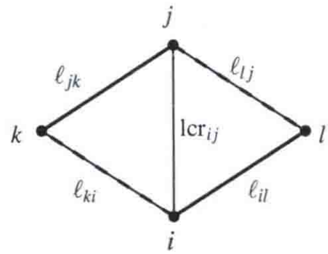

Fig. 3 Length cross-ratio

implicitly assumes that an orientation has been chosen on the surface. For non-orientable surfaces, the length cross-ratio is well-defined on the oriented double cover.

The product of length cross-ratios around an interior vertex $i \in V$ is 1, because all lengths cancel:

$$\prod_{ij \ni i} \mathrm{lcr}_{ij} = 1. \tag{10}$$

Proposition 2.6 *Two euclidean triangulations $(\Sigma, \ell)_{euc}$ and $(\Sigma, \tilde{\ell})_{euc}$ are discretely conformally equivalent if and only if for each interior edge $ij \in E_{\Sigma}^{int}$, the induced length cross-ratios agree.*

Remark 2.7 Analogous statements hold for spherical and hyperbolic triangulations. Equation (9) has to be modified by replacing ℓ with $\sin \frac{\ell}{2}$ or $\sinh \frac{\ell}{2}$, respectively (compare Remark 2.3).

2.6 *Triangulations: Reconstructing Lengths from Length Cross-Ratios*

To deal with Riemann surfaces that are given in terms of Schottky data (Sect. 8.2) we will need to reconstruct a function $\ell : E_{\Sigma} \to \mathbb{R}_{>0}$ satisfying (9) from given length cross-ratios. (It is not required that the function ℓ satisfies the triangle inequalities.) To this end, we define auxiliary quantities c^i_{jk} attached to the angles of the triangulation. The value at vertex i of the triangle $ijk \in F$ is defined as

$$c^i_{jk} = \frac{\ell_{jk}}{\ell_{ij}\ell_{ki}}. \tag{11}$$

Then (9) is equivalent to

$$\mathrm{lcr}_{ij} = \frac{c^i_{jk}}{c^i_{lj}}. \tag{12}$$

Now, given a function $\mathrm{lcr} : E^{int} \to \mathbb{R}_{>0}$ defined on the set of interior edges E^{int} and satisfying the product condition (10) around interior vertices, one can find parame-