Martingales in Banach Spaces

GILLES PISIER

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GILLES PISIER
Texas A&M University

CAMBRIDGE
UNIVERSITY PRESS

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This book is focused on the major applications of martingales to the geometry of Banach spaces, but a substantial discussion of harmonic analysis in Banach space valued Hardy spaces is presented. Exciting links between super-reflexivity and some metric spaces related to computer science are covered, as is an outline of the recently developed theory of non-commutative martingales, which has natural connections with quantum physics and quantum information theory.

Requiring few prerequisites and providing fully detailed proofs for the main results, this self-contained study is accessible to graduate students with basic knowledge of real and complex analysis and functional analysis. Chapters can be read independently, each building from introductory notes, and the diversity of topics included also means this book can serve as the basis for a variety of graduate courses.

Gilles Pisier was a professor at the University of Paris VI from 1981 to 2010 and has been Emeritus Professor since then. He has been a distinguished professor and holder of the Owen Chair in Mathematics at Texas A&M University since 1985. His international prizes include the Salem Prize in harmonic analysis (1979), the Ostrowski Prize (1997), and the Stefan Banach Medal (2001). He is a member of the Paris Académie des sciences, a Foreign Member of the Polish and Indian Academies of Science, and a Fellow of both the IMS and the AMS. He is also the author of several books, notably *The Volume of Convex Bodies and Banach Space Geometry* (1989) and *Introduction to Operator Space Theory* (2002), both published by Cambridge University Press.

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Introduction

Martingales (with discrete time) lie at the centre of this book. They are known to have major applications to virtually every corner of probability theory. Our central theme is their applications to the geometry of Banach spaces.

We should emphasize that we do *not* assume any knowledge about scalar valued martingales. Actually, the beginning of this book gives a self-contained introduction to the basic martingale convergence theorems for which the use of the norm of a vector valued random variable instead of the modulus of a scalar one makes little difference. Only when we consider the 'boundedness implies convergence' phenomenon does it start to matter. Indeed, this requires the Banach space B to have the Radon-Nikodým property (RNP). But even at this point, the reader who wishes to concentrate on the scalar case could simply assume that B is finite-dimensional and disregard all the infinite-dimensional technical points. The structure of the proofs remains pertinent if one does so. In fact, it may be good advice for a beginner to do a first reading in this way. One could argue similarly about the property of 'unconditionality of martingale differences' (UMD): although perhaps the presence of a Banach space norm is more disturbing there, our reader could assume at first reading that B is a Hilbert space, thus getting rid of a number of technicalities to which one can return later.

A major feature of the UMD property is its equivalence to the boundedness of the Hilbert transform (HT). Thus we include a substantial excursion in (Banach space valued) harmonic analysis to explain this.

Actually, connections with harmonic analysis abound in this book, as we include a rather detailed exposition of the boundary behaviour of B-valued harmonic (resp. analytic) functions in connections with the RNP (resp. analytic RNP) of the Banach space B. We introduce the corresponding B-valued Hardy spaces in analogy with their probabilistic counterparts. We are partly motivated

by the important role they play in operator theory, when one takes for B the space of bounded operators (or the Schatten p-class) on a Hilbert space.

Hardy spaces are closely linked with martingales via Brownian motion: indeed, for any B-valued bounded harmonic (resp. analytic) function u on the unit disc D, the composition $(u(W_{t \wedge T}))_{t>0}$ of u with Brownian motion stopped before it exits D is an example of a continuous B-valued martingale, and its boundary behaviour depends in general on whether B has the RNP (resp. analytic RNP). We describe this connection with Brownian motion in detail, but we refrain from going too far on that road, remaining faithful to our discrete time emphasis. However, we include short sections summarizing just what is necessary to understand the connections with Brownian martingales in the Banach valued context, together with pointers to the relevant literature. In general, the sections that are a bit far off our main goals are marked by an asterisk. For instance, we describe in §7.1 the Banach space valued version of Fefferman's duality theorem between H^1 and BMO. While this is not really part of martingale theory, the interplay with martingales, both historically and heuristically, is so obvious that we felt we had to include it. The asterisked sections could be kept for a second reading.

In addition to the RN and UMD properties, our third main theme is super-reflexivity and its connections with uniform convexity and smoothness. Roughly, we relate the geometric properties of a Banach space B with the study of the p-variation

$$S_p(f) = \left(\sum_1^\infty \| f_n - f_{n-1} \|_B^p \right)^{1/p}$$

of B-valued martingales (f_n). Depending on whether $S_p(f) \in L_p$ is necessary or sufficient for the convergence of (f_n) in $L_p(B)$, we can find an equivalent norm on B with modulus of uniform convexity (resp. smoothness) 'at least as good as' the function $t \to t^p$.

We also consider the strong p-variation

$$V_p(f) = \sup_{0=n(0)<n(1)<n(2)<\cdots} \left(\sum_1^\infty \| f_{n(k)} - f_{n(k-1)} \|_B^p \right)^{1/p}$$

of a martingale. For that topic (exceptionally) we devote an entire chapter only to the scalar case. Our crucial tool here is the 'real interpolation method'. Real and complex interpolation in general play an important role in L_p-space theory, so we find it natural to devote a significant amount of space to these two 'methods'.

We allow ourselves several excursions aiming to illustrate the efficiency of martingales, for instance to the concentration of measure phenomenon. We also

describe some exciting recent work on non-linear properties of metric spaces analogous to uniform convexity/smoothness and type for metric spaces.

We originally intended to include in this book a detailed presentation of 'non-commutative' martingale theory, but that part became so big that we decided to make it the subject of a (hopefully forthcoming) separate volume to be published, perhaps on the author's web page. We merely outline its contents in the last chapter, devoted to non-commutative L_p-spaces. There the complex interpolation method becomes a central tool.

The book should be accessible to graduate students, requiring only the basics of real and complex analysis (mainly Lebesgue integration) and basic functional analysis (mainly duality, the weak and strong topologies and reflexivity of Banach spaces). Our choice is to give fully detailed proofs for the main results and to indicate references to the refinements in the 'Notes and Remarks' or the asterisked sections. We strive to make the presentations self-contained, and when given a choice, we opt for simplicity over maximal generality. For instance, we restrict the Banach space valued harmonic analysis to functions with domains in the unit disc D or the upper half-plane U in $\mathbb{C}$ (or their boundary $\partial D = \mathbb{T}$ or $\partial U = \mathbb{R}$). We feel the main ideas are easier to grasp in the real or complex uni-dimensional case.

The topics (martingales, H^p-space theory, interpolation, Banach space geometry) are quite diverse and should appeal to several distinct audiences. The main novelty is the choice to bring all these topics together in the various parts of this single volume. We should emphasize that the different parts can be read independently, and each time their start is introductory.

There are natural groupings of chapters, such as 1-2-10-11 or 3-4-5-6 (possibly including parts of 1 and 2, but not necessarily), which could form the basis for a graduate course.

Depending on his or her background, a reader is likely to choose to concentrate on different parts. We hope probabilist graduate students will benefit from the detailed introductory presentation of basic H^p-space theory, its connections with martingales, the links with Banach space geometry and the detailed treatment of interpolation theory (which we illustrate by applications to the strong p-variation of martingales), while graduate students with interest in functional analysis and Banach spaces should benefit more from the initial detailed presentation of basic martingale theory. In addition, we hope to attract readers with interest in computer science wishing to see the sources of the various recent developments on finite metric spaces described in Chapter 13. A reader with an advanced knowledge of harmonic analysis and H^p-theory will probably choose to skip the introductory part on that direction, which is written

with non-specialists in mind, and concentrate on the issues specific to Banach space valued functions related to the UMD property and the Hilbert transform.

The choice to include so much background on the real and complex interpolation methods in Chapter 8 is motivated by its crucial importance in Banach space valued L_p-space theory, which, in some sense, is the true subject of this book.

Description of the contents

We will now review the contents of this book chapter by chapter.

Chapter 1 begins with preliminary background: we introduce Banach space valued L_p-spaces, conditional expectations and the central notion in this book, namely Banach space valued martingales associated to a filtration $(\mathcal{A}_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We describe the classical examples of filtrations (the dyadic one and the Haar one) in §1.4. If B is an arbitrary Banach space and the martingale (f_n) is associated to some f in $L_p(B)$ by $f_n = \mathbb{E}^{\mathcal{A}_n}(f)$ ($1 \leq p < \infty$), then, assuming $\mathcal{A} = \mathcal{A}_\infty$ for simplicity, the fundamental convergence theorems say that

$$f_n \to f$$

both in $L_p(B)$ and almost surely (a.s.).

The convergence in $L_p(B)$ is Theorem 1.14, while the a.s. convergence is Theorem 1.30. The latter is based on Doob's classical *maximal inequalities* (Theorem 1.25), which are proved using the crucial notion of *stopping time*. We also describe the dual form of Doob's inequality due to Burkholder-Davis-Gundy (see Theorem 1.26). Doob's maximal inequality shows that the convergence of f_n to f in $L_p(B)$ 'automatically' implies a.s. convergence. This, of course, is special to martingales, but in general it requires $p \geq 1$. However, for martingales that are sums of independent, symmetric random variables (Y_n) (i.e. we have $f_n = \sum_1^n Y_k$), this result holds for $0 < p < 1$ (see Theorem 1.40). It also holds, roughly, for $p = 0$. This is the content of the celebrated Ito-Nisio theorem (see Theorem 1.43), which asserts that even a weak form of convergence of the series $f_n = \sum_1^n Y_k$ implies its a.s. norm convergence.

In §1.8, we prove, again using martingales, a version of Phillips's theorem. The latter is usually stated as saying that, if B is separable, any countably additive measure on the Borel σ-algebra of B is 'Radon', i.e. the measure of a Borel

subset can be approximated by that of its compact subsets. In §1.9, we prove the strong law of large numbers using the a.s. convergence of reverse B-valued martingales. In §1.10, we give a brief introduction to continuous-time martingales. We mainly explain the basic approximation technique by which one passes from discrete to continuous parameter.

To get to a.s. convergence, all the preceding results need to assume in the first place some form of convergence, e.g. in $L_p(B)$. In classical (i.e. real valued) martingale theory, it suffices to assume *boundedness* of the martingale $\{f_n\}$ in L_p ($p \geq 1$) to obtain its a.s. convergence (as well as norm convergence, if $1 < p < \infty$). However, this 'boundedness $\Rightarrow$ convergence' phenomenon no longer holds in the B-valued case unless B has a specific property called the Radon-Nikodým property (RNP in short), which we introduce and study in Chapter 2. The RNP of a Banach space B expresses the validity of a certain form of the Radon-Nikodým theorem for B-valued measures, but it turns out to be equivalent to the assertion that all martingales bounded in $L_p(B)$ converge a.s. (and in $L_p(B)$ if $p > 1$) for some (or equivalently all) $1 \leq p < \infty$. Moreover, the RNP is equivalent to a certain 'geometric' property called 'dentability'. All this is included in Theorem 2.9. The basic examples of Banach spaces with the RNP are the reflexive ones and separable duals (see Corollary 2.15).

Moreover, a dual space B^* has the RNP iff the classical duality $L_p(B)^* = L_{p'}(B^*)$ is valid for some (or all) $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ (see Theorem 2.22). Actually, for a general B, one can also describe $L_p(B)^*$ as a space of martingales bounded in $L_{p'}(B^*)$, but in general, the latter is larger than the (Bochner sense) space $L_{p'}(B^*)$ itself (see Proposition 2.20). In many situations, it is preferable to have a description of $L_p(B)^*$ as a space of B^*-valued measures or functions (rather than martingales). For that purpose, we give in §2.4 two alternate descriptions of the latter space, either as a space of B^*-valued vector measures, denoted by $\Lambda_{p'}(B^*)$, or, assuming B separable, as a space of weak* measurable B^*-valued functions denoted by $\underline{\Lambda}_{p'}(B^*)$.

In §2.5, we discuss the Krein-Milman property (KMP): this says that any bounded closed convex set $C \subset B$ is the closed convex hull of its extreme points. This is closely related to dentability, but although it is known that RNP $\Rightarrow$ KMP (see Theorem 2.34), the converse implication is still open.

In §2.6, we present a version of Choquet's representation theorem of the points of C as barycenters of measures supported by the extreme points. Choquet's classical result applies only to convex compact sets, while this version requires that the closed bounded separable convex set C lies in an RNP space. The proof is based on martingale convergence.

In §2.7, we prove that, if B is separable, a bounded B-valued vector measure admits a RN density iff the associated linear operator $L_1 \to B$ factorizes

through the space ℓ_1 of absolutely summable scalar sequences. This fact (due to Lewis and Stegall) is remarkable because ℓ_1 is the prototypical example of a separable dual (a fortiori ℓ_1 has the RNP). In other words, if B has the RNP, the bounded B-valued vector measures 'come from' ℓ_1 valued vector measures, and the latter are differentiable since ℓ_1 itself has the RNP. There is also a version of this result when B is not separable (see Theorem 2.39).

In Chapter 3, we introduce the Hardy space $h^p(D; B)$ $(1 \le p \le \infty)$ formed of all the B-valued harmonic functions $u: D \to B$ (on the unit disc $D \subset \mathbb{C}$) such that

$$\|u\|_{h^p(D;B)} = \sup_{0<r<1} \left(\int \|u(re^{it})\|^p dm(t) \right)^{1/p} < \infty.$$

When $p = \infty$, this is just the space of bounded harmonic functions $u: D \to B$.

In analogy with the martingale case treated in Chapter 1, to any f in $L_p(\mathbb{T}; B)$, we can associate a harmonic function $u: D \to B$ that extends f in the sense that $u(re^{it}) \to f(e^{it})$ for almost all e^{it} in $\partial D = \mathbb{T}$, and also (if $p < \infty$)

$$\int \|u(re^{it}) - f(e^{it})\|^p dm(t) \to 0$$

when $r \uparrow 1$ (see Theorem 3.1). The convergence at almost all boundary points requires a specific radial maximal inequality, which we derive from the classical Hardy-Littlewood maximal inequality. Actually, we present this in the framework of 'non-tangential' maximal inequalities. The term 'non-tangential' refers to the fact that we study the limit of $u(z)$ (in the norm of B) when z tends to e^{it} but staying inside a cone with vertex e^{it} and opening angle $\beta < \frac{\pi}{2}$.

This topic is closely linked with Brownian motion (see especially [176]). Indeed, the paths of a complex valued Brownian motion $(W_t)_{t>0}$ (starting at the origin) almost surely cross the boundary of the unit disc D in finite time, and if we define $T_r = \inf\{t > 0 | |W_t| = r\}$, then for any u in $h^p(D; B)$, the random variables $\{u(W_{T_r}) \mid 0 < r < 1\}$ form a martingale bounded in $L_p(B)$, for which the maximal function is closely related to the non-tangential one of u.

In general, the latter B-valued martingales do not converge. However, we show in Corollary 3.31 that they do so if B has the RNP. Actually, the RNP of B is equivalent to the a.e. existence of radial (or non-tangential) limits or of limits along almost all Brownian paths, for the functions in $h^p(D; B)$ $(1 \le p \le \infty)$ (see Theorem 3.25 and Corollary 3.31 for this). In the supplementary §4.7, we compare the various modes of convergence to the boundary, radial, non-tangential or Brownian for a general B-valued harmonic function. This brief §4.7 outlines some beautiful work by Burkholder, Gundy and Silverstein [176] and Brossard [148].

In Chapter 4 we turn to the subspace $H^p(D; B) \subset h^p(D; B)$ formed of all the *analytic B*-valued functions in $h^p(D; B)$. One major difference is that the (radial or non-tangential) maximal inequalities now extend to all values of p including $0 < p \leq 1$. To prove this, we make crucial use of *outer functions*; this gives us a convenient factorization of any f in $H^p(D; B)$ as a product $f = Fg$ with a *scalar* valued F in H^p and g in $H^\infty(D; B)$ (see Theorem 4.15). The Hardy spaces $H^p(D; B)$ naturally lead us to a more general form of RNP called the analytic RNP (ARNP in short). This is equivalent to the a.e. existence of radial (or non-tangential) limits for all functions in $H^p(D; B)$ for some (or all) $0 < p \leq \infty$. Thus we have RNP $\Rightarrow$ ARNP, but the converse fails, for instance (see Theorem 4.32) $B = L_1([0, 1])$ has the ARNP but not the RNP. On the martingale side, the strict inclusion

$$\{\text{analytic}\} \subsetneq \{\text{harmonic}\}$$

admits an analogue involving the notion of 'analytic martingale' or 'Hardy martingale', and the convergence of the latter (with the usual bounds) is equivalent to the ARNP (see Theorem 4.30). In §4.6 we briefly review the analogous Banach space valued H^p-space theory for functions on the upper half-plane

$$U = \{z \in \mathbb{C} | \Im(z) > 0\}.$$

Chapter 5 is devoted to the UMD property. After a brief presentation of Burkholder's inequalities in the scalar case, we concentrate on their analogue for Banach space valued martingales (f_n). In the scalar case, when $1 < p < \infty$, we have

$$\sup_n \|f_n\|_p \simeq \| \sup |f_n| \|_p \simeq \|S(f)\|_p,$$

where $S(f) = (|f_0|^2 + \sum |f_n - f_{n-1}|^2)^{1/2}$, and where $A_p \simeq B_p$ means that there are positive constants C'_p and C''_p such that $C'_p A_p \leq B_p \leq C''_p A_p$. In the Banach space valued case, we replace $S(f)$ by

$$R(f)(\omega) = \sup_N \left(\int \left\| f_0(\omega) + \sum_1^N \varepsilon_n(f_n - f_{n-1})(\omega) \right\|^2 dv \right)^{1/2}, \quad (1)$$

where v is the uniform probability measure on the set Δ of all choices of signs $(\varepsilon_n)_n$ with $\varepsilon_n = \pm 1$.

In §5.2 we prove Kahane's inequality, i.e. the equivalence of all the L_p-norms for series of the form $\sum_1^\infty \varepsilon_n x_n$ with x_n in an arbitrary Banach space when $0 < p < \infty$ (see (5.16)); in particular, up to equivalence, we can substitute to the L_2-norm in (1) any other L_p-norm for $p < \infty$.

Let $\{x_n\}$ be a sequence in a Banach space such that the series $\sum \varepsilon_n x_n$ converges almost surely. We set

$$R(\{x_n\}) = \left(\int_D \left\| \sum \varepsilon_n x_n \right\|^2 dv \right)^{1/2}.$$

With this notation we have

$$R(f)(\omega) = R(\{f_0(\omega), f_1(\omega) - f_0(\omega), f_2(\omega) - f_1(\omega), \cdots \}). \tag{2}$$

The UMD_p and UMD properties are introduced in §5.3. Consider the series

$$\tilde{f}_\varepsilon = f_0 + \sum_1^\infty \varepsilon_n (f_n - f_{n-1}). \tag{3}$$

By definition, when B is UMD_p, (f_n) converges in $L_p(B)$ iff (3) converges in $L_p(B)$ for all choices of signs $\varepsilon_n = \pm 1$ or equivalently iff it converges for almost all (ε_n). Moreover, we have then for $1 < p < \infty$ and all choices of signs $\varepsilon = (\varepsilon_n)$

$$\|\tilde{f}_\varepsilon\|_{L_p(B)} \simeq \|f\|_{L_p(B)} \tag{4}_p$$

$$\sup_{n \geq 0} \|f_n\|_{L_p(B)} \simeq \|R(f)\|_p. \tag{5}_p$$

See Proposition 5.10. The case $p = 1$ (due to Burgess Davis) is treated in §5.6. The main result of §5.3 is the equivalence of UMD_p and UMD_q for any $1 < p, q < \infty$. We give two proofs of this; the first one is based on distributional (also called 'good λ') inequalities. This is an extrapolation principle, which allows us to show that, for a given Banach space B, $(4)_q \Rightarrow (4)_p$ for any $1 < p < q$. In the scalar case one starts from the case $q = 2$, which is obvious by orthogonality, and uses the preceding implication to deduce from it the case $1 < p < 2$ and then $2 < p < \infty$ by duality. We also give a more delicate variant of the extrapolation principle that avoids duality and deduces the desired inequality for any $1 < p < \infty$ from a certain form of weak-type estimate involving pairs of stopping times (see Lemma 5.26).

The second proof is based on Gundy's decomposition, which is a martingale version of the Calderón-Zygmund decomposition in classical harmonic analysis. There one proves a weak-type (1,1) estimate and then invokes the Marcinkiewicz theorem to obtain the case $1 < p < 2$. We describe the latter in an appendix to Chapter 5.

In §5.8 we show that to check that a space B is UMD_p, we may restrict ourselves to martingales adapted to the *dyadic* filtration, and the associated UMD-constant remains the same. The proof is based on a result of independent interest: if $p < \infty$, any finite martingale $(f_0, \ldots, f_N)$ in $L_p(B)$ (on a large enough

probability space) can be approximated in $L_p(B)$ by a subsequence of a dyadic martingale.

In §5.9 we prove the Burkholder-Rosenthal inequalities. In the scalar case, this boils down to the equivalence

$$\sup_n \|f_n\|_p \simeq \|\sigma(f)\|_p + \|\sup_n |f_n - f_{n-1}|\|_p,$$

where $\sigma(f) = (|f_0|^2 + \sum \mathbb{E}_{n-1}|f_n - f_{n-1}|^2)^{1/2}$, valid for $2 \leq p < \infty$.

Rosenthal originally proved this when f_n is a sum of independent variables, and Burkholder extended it to martingales. We describe a remarkable example of complemented subspace of L_p (the Rosenthal space X_p), which motivated Rosenthal's work.

In §5.10 we describe Stein's inequality and its B-valued analogue when B is a UMD Banach space. Let $(\mathcal{A}_n)_{n \geq 0}$ be a filtration as usual, and let $(x_n)_{n \geq 0}$ be now an *arbitrary* sequence in L_p. Let $y_n = \mathbb{E}^{\mathcal{A}_n} x_n$. Stein's inequality asserts that for any $1 < p < \infty$, there is a constant C_p such that

$$\left\|\left(\sum |y_n|^2\right)^{1/2}\right\|_p \leq C_p \left\|\left(\sum |x_n|^2\right)^{1/2}\right\|_p \tag{6}$$

for any (x_n) in L_p.

For x_n in $L_p(B)$, with B UMD, the same result remains valid if we replace on both sides of (6) the expression $(\sum |x_n|^2)^{1/2}$ by

$$\left(\int \left\|\sum \varepsilon_n x_n\right\|_B^2 dv\right)^{1/2}.$$

See (5.89).

In §5.11 we describe Burkholder's geometric characterization of UMD spaces in terms of ζ-convexity (Theorem 5.64). We also include a more recent result (Theorem 5.69) in the same vein. The latter asserts that a real Banach space of the form $B = X \oplus X^*$ is UMD iff the function

$$x \oplus \xi \to \xi(x)$$

is the difference of two real valued convex continuous functions on B.

We end Chapter 5 by a series of appendices. In §5.12 we prove the hypercontractive inequalities on $\{-1, 1\}$. In §5.13 we discuss the Hölder-Minkowski inequality, which says that we have a norm 1 inclusion

$$L_q(m'; L_p(m)) \subset L_p(m; L_q(m'))$$

when $p \geq q$. In §5.14 we give some basic background on the space weak-L_p, usually denoted $L_{p,\infty}$, and §5.16 is devoted to a quick direct proof of

the Marcinkiewicz theorem. In §5.15 we present a trick frequently used by Burkholder that we call the reverse Hölder principle. Hölder's inequality for a random variable Z on a probability space tells us that $Z \in L_q \Rightarrow Z \in L_r$ when $r < q$. The typical reverse Hölder principle shows that a suitable L_r bound involving independent copies of Z implies conversely that Z is in weak-L_p (or $L_{p,\infty}$) and a fortiori in L_q for all $r < q < p$.

In the final appendix to this chapter, §5.17, we explain why certain forms of exponential integrability of a function f can be equivalently reformulated in terms of the growth of the L_p-norms of f. This shows that the growth when $p \to \infty$ of the constants in many of the martingale inequalities we consider often implies an exponential inequality.

In Chapter 6 we start with some background on the Hilbert transform. We say that a Banach space B is an HT-space if the Hilbert transform on $\mathbb{T}$ defines a bounded operator on $L_p(\mathbb{T}; B)$ for some (or equivalently all) $1 < p < \infty$. We then show that UMD and HT are equivalent properties. To show HT $\Rightarrow$ UMD, we follow Bourgain's well-known argument (see §6.2). The converse implication UMD $\Rightarrow$ HT was originally proved using ideas derived from the beautiful observation that the Hilbert transform can be viewed as a sort of martingale transform relative to stochastic integrals over Brownian motion. We merely outline this proof in §6.4, and instead, we present first, in full details the more recent remarkable proof from Petermichl [378], which uses only martingale transforms relative to Haar systems, but with respect to a randomly chosen dyadic filtration (see §6.3).

In §6.5, following Bourgain, we prove that the Littlewood-Paley inequalities are valid in the B-valued case if B is UMD. More precisely, consider the formal Fourier series

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int}$$

of a function in $L_p(B)$, and let

$$V(f) = (\|\hat{f}(0)\|_B^2 + R(\{\Delta_n^+\})^2 + R(\{\Delta_n^-\})^2)^{1/2},$$

where

$$\Delta_n^+ = \sum_{2^n \leq k < 2^{n+1}} \hat{f}(k) e^{int} \quad \text{and} \quad \Delta_n^- = \sum_{2^n \leq -k < 2^{n+1}} \hat{f}(k) e^{ikt}.$$

The B-valued version of the Littlewood-Paley inequality is the equivalence of $\|f\|_{L_p(B)}$ and $\|V(f)\|_p$.

In Theorem 6.35 we show that the analogue of the Hilbert transform for the compact group $\{-1, 1\}^{\mathbb{N}}$ (in place of $\mathbb{T}$) is bounded on $L_p(B)$ for any $1 < p < \infty$ iff B is UMD. This is the analogue of the implication UMD $\Rightarrow$ HT, but for

the Walsh system in place of the trigonometric one. The proof of this case is much more transparent.

In §6.7, we briefly review the analytic UMD property (AUMD). This is a weakening of UMD obtained by restricting to either analytic or Hardy martingales. The latter are discretizations of the B-valued martingales obtained by composing complex Brownian motion with a B-valued analytic function on D. The main novelty is that the space $B = L_1$ itself has the AUMD property. Moreover, the latter implies the ARNP.

In §7.1 we describe the B-valued version of Fefferman's famous duality theorem between H^1 and *BMO* on $\mathbb{T}$ (resp. $\mathbb{R}$). This requires that we carefully identify the various B-valued analogues of the Hardy spaces H^1 or *BMO*. If B is arbitrary, the duality holds provided that we use the atomic version of B-valued H^1, denoted by $h^1_{at}(\mathbb{T}; B)$ (resp. $h^1_{at}(\mathbb{R}; B)$); this is the so-called real variable variant of H^1, as opposed to the more classical Hardy space of analytic function, denoted by $H^1(D; B)$ (resp. $H^1(U; B)$) on the unit disc D (resp. the upper half-plane U). In general, for any $f \in H^1(D; B)$ (resp. $H^1(U; B)$), the boundary values of f and its Hilbert transform $\tilde{f}$ are both in $h^1_{at}(\mathbb{T}; B)$ (resp. $h^1_{at}(\mathbb{R}; B)$). When B is UMD (and only then), the converse also holds (see Corollary 7.20).

In §7.3 we discuss the space BMO and the B-valued version of H^1 in the martingale context. This leads naturally to the atomic version of B-valued H^1, denoted by $h^1_{at}(\{\mathcal{A}_n\}; B)$ with respect to a filtration $\{\mathcal{A}_n\}$. Its dual can be identified with a BMO-space for B^*-valued martingales, at least for a 'regular' filtration $(\mathcal{A}_n)$. Equivalently (see Theorem 7.32), the space $h^1_{at}(B)$ can be identified with $\tilde{h}^1_{max}(B)$, which is defined as the completion of $L_1(B)$ with respect to the norm $f \mapsto \mathbb{E} \sup_n \|f_n\|_B$ (here $f_n = \mathbb{E}^{\mathcal{A}_n} f$).

In §7.5 we show that the classical BMO space coincides with the intersection of two suitably chosen translates of the dyadic martingale-BMO space.

In Chapter 8 we describe successively the complex and the real method of interpolation for pairs of Banach spaces (B_0, B_1) assumed compatible for interpolation purposes. The complex interpolation space is denoted by $(B_0, B_1)_\theta$. It depends on the single parameter $0 < \theta < 1$ and requires B_0, B_1 to be both *complex* Banach spaces. Complex interpolation is a sort of 'abstract' generalization of the classical Riesz-Thorin theorem, asserting that if an operator T has norm 1 simultaneously on both spaces $B_0 = L_{p_0}$ and $B_1 = L_{p_1}$, with $1 \le p_0 < p_1 \le \infty$, then it also has norm 1 on the space L_p for any p such that $p_0 < p < p_1$. Since we make heavy use of this *complex method* in noncommutative L_p-theory, we review its basic properties somewhat extensively.

The real interpolation space is denoted by $(B_0, B_1)_{\theta,q}$. It depends on two parameters $0 < \theta < 1$, $1 \le q \le \infty$, and now (B_0, B_1) can be a pair of *real* Banach spaces. Real interpolation is a sort of abstract generalization of the