Introduction to Algebraic *K*-Theory

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LONDON AND NEW YORK
CHAPMAN AND HALL

First published 1981 by
Chapman and Hall Ltd
11 New Fetter Lane, London EC4P 4EE
Published in the USA by
Chapman and Hall
in association with Methuen, Inc.
733 Third Avenue, New York NY 10017

Printed in Great Britain by
Whitstable Litho Ltd., Whitstable, Kent

ISBN 0 412 22700 2 (cased)
ISBN 0 412 23740 7 (paperback)

British Library Cataloguing in Publication Data

Silvester, John R.
Introduction to algebraic *K*-theory.
— (Chapman and Hall mathematics series)
1. Algebra, Linear
I. Title
512′.5 QA184

ISBN 0-412-22700-2
ISBN 0-412-23740-7 Pbk

PREFACE

Algebraic K-theory is the name given to a body of theory which may be regarded in the first instance as an attempt to generalize parts of linear algebra, notably the theory of dimension of vector spaces, and determinants, to modules over arbitrary rings. The subject gets its name from its notation - given a ring R, one constructs groups called K_0R, K_1R, K_2R, ... - and is called algebraic K-theory, rather than just K-theory, because it derives from (topological) K-theory, which is to do with the topology of vector bundles.

The intention of this text is to make algebraic K-theory accessible at a more elementary level than heretofore. The only absolute prerequisites are standard undergraduate first courses in linear algebra and in groups and rings, although an acquaintance with the beginnings of the theory of group presentations, modules, and categories, would be helpful. From time to time, some algebraic topology is used, but these sections can easily be omitted at a first reading.

I have tried to make the text as self-contained as possible, so I have included proofs of many standard results on such topics as, for example, tensor products and modules of fractions.

I have also reduced to a minimum the number of proofs left to the reader; at the risk of tedium, most proofs are given in full detail. The more sophisticated reader is encouraged to skip the proofs (or, better, provide his own) whenever the propositions seem obvious.

The text is an expanded version of a London M.Sc. lecture course given at King's College in 1976. The approach inevitably owes much to the standard texts by Bass, Milnor, and Swan (listed in the Bibliography). The choice of material is somewhat arbitrary, being limited by space and by the requirement that it be elementary, and so I have been content to establish the most basic properties of the functors K_0, K_1, K_2, and to do a few explicit computations. Many important topics, such as the K-theory of polynomial extensions, and localization exact sequences, are omitted; Quillen's higher K-theory is also beyond our scope.

My thanks are due to many people for their interest and encouragement, and I wish to thank especially Keith Dennis and Michael Stein for conversations and correspondence which have strongly influenced later sections of the text. I wish also to thank the students and colleagues who attended the original course of lectures, especially Philip Higgins and Tony Barnard, whose penetrating questions and insistence on clarity were a constant stimulus.

London J.R.S.

March, 1981

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C H A P T E R   O N E

Modules

The word *ring* will mean an associative, but not necessarily commutative, ring with a multiplicative identity, written 1. Let R, S be rings. A map $f : R \to S$ is a (ring) *homomorphism* if, for all $r, s \in R$, we have $f(r + s) = f(r) + f(s)$ and $f(rs) = f(r)f(s)$, and also $f(1) = 1$. Note that in this last equation the symbol 1 has two meanings: on the left, $1 \in R$, and on the right, $1 \in S$.

Let R be a ring. A (*left*) *R-module* is an abelian group M, written additively, with a map $R \times M \to M$, called *scalar multiplication* and written $(r, m) \mapsto rm$ ($r \in R$, $m \in M$), such that, for all $r, s \in R$ and $m, n \in M$, we have

$$r(m + n) = rm + rn \qquad \text{(i)}$$

$$(r + s)m = rm + sm \qquad \text{(ii)}$$

$$(rs)m = r(sm) \qquad \text{(iii)}$$

$$1m = m. \qquad \text{(iv)}$$

Elements of R are then referred to as *scalars*. A *right* R-module is defined similarly, except that (iii) is replaced by

$$(rs)m = s(rm). \qquad \text{(iii')}$$

Alternatively, and more naturally, the scalars may be written on the right, so that (iii') reads $m(sr) = (ms)r$. The word *module* will usually mean a *left* module. Of course, if the ring R is commutative, the notions of left and right R-module coincide; if R is a field, an R-module is just a vector space over R. If $R = \mathbb{Z}$, the ring of rational integers, an R-module is just an (additive) abelian group.

Let M be an R-module. Remember M is a group: a subgroup N of M is called a *submodule* if it is closed under scalar multiplication, that is, if $rn \in N$ for all $r \in R$ and $n \in N$. More generally, if N is any non-empty subset of M, write RN for the set of all finite sums $\Sigma_i r_i n_i$ $(r_i \in R, n_i \in N)$. RN is clearly a submodule of M, and is the smallest submodule of M containing N. Thus N is a submodule if and only if $RN = N$. Obviously M itself is a submodule, and so is $\{0\}$, where 0 denotes the additive identity of M. This follows from the fact that $r0 = 0$, all $r \in R$, which is easily deduced from the module axioms.

Let N be a submodule of M. The quotient group M/N becomes an R-module if we define the scalar multiplication by

$$r(m + N) = rm + N \quad (r \in R, m \in M).$$

It is easy to see that this is well-defined and satisfies the axioms. M/N is then called the *quotient module* of M by N.

Let M, N be R-modules. A map $f : M \to N$ is called a (module) *homomorphism* if, for all $m, n \in M$ and $r \in R$, we have

$$f(m + n) = f(m) + f(n)$$

and

$$f(rm) = r[f(m)].$$

The *kernel*

$$\ker f = f^{-1}\{0\} = \{m \in M : f(m) = 0\}$$

is a submodule of M, and the *image*

$$f(M) = \{f(m) : m \in M\}$$

is a submodule of N. The homomorphism f is a *monomorphism* if it is an injection, or equivalently if $\ker f = 0$; here we are writing 0 for the zero (sub)module: $0 = \{0\}$. The homomorphism f is an *epimorphism* if it is a surjection, that is, if $f(M) = N$. We sometimes write $f : M \hookrightarrow N$ for a monomorphism and $f : M \twoheadrightarrow N$

for an epimorphism. If f is both a monomorphism and an epimorphism, it is an *isomorphism*; if such an f exists we say M, N are *isomorphic*, and write $M \cong N$. In this case, $f^{-1} : N \to M$ is also an isomorphism. An *endomorphism* is a homomorphism $M \to M$, and an *automorphism* is an isomorphism $M \to M$. The identity map $M \to M$ is an automorphism. The *first isomorphism theorem* states that if $f : M \to N$ is a homomorphism then $M/\ker f \cong f(M)$. The proof is left to the reader.

1.1 *Direct sums*

Let M be an R-module, with submodules M_1, M_2. If every element of M can be written uniquely in the form $m_1 + m_2$ ($m_1 \in M_1$, $m_2 \in M_2$) we say M is the *direct sum* of M_1 and M_2, and write $M = M_1 \oplus M_2$. If, for arbitrary submodules M_1, M_2 of M we write

$$M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}$$

then it is clear that $M_1 + M_2$ and $M_1 \cap M_2$ are submodules of M, and that $M = M_1 \oplus M_2$ if and only if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$.

More generally, if M is an R-module with submodules M_λ ($\lambda \in \Lambda$), where the index set Λ may be infinite, and if every element $m \in M$ can be written uniquely, except for zeros and the order of the terms, as a finite sum $m = \Sigma_\lambda m_\lambda$ ($m_\lambda \in M_\lambda$), then we say M is the *direct sum* of the M_λ, and write $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$.

Now suppose we are given R-modules M_λ ($\lambda \in \Lambda$). We shall construct their direct sum. Let M be the subset of the cartesian product $\times_{\lambda\in\Lambda} M_\lambda$ consisting of all Λ-tuples (m_λ) with $m_\lambda = 0$ for almost all λ (that is, for all but a finite number of values of λ). M becomes an R-module if we define addition and scalar multiplication componentwise; explicitly,

$$(m_\lambda) + (n_\lambda) = (m_\lambda + n_\lambda) \text{ and } r(m_\lambda) = (rm_\lambda)$$

where m_λ, $n_\lambda \in M_\lambda$ and $r \in R$. For each $\lambda \in \Lambda$, M contains a submodule

$$M'_\lambda = \{(m_\mu) : m_\mu = 0 \text{ for all } \mu \neq \lambda\}$$

which is isomorphic to M_λ. If we identify M_λ with M'_λ in the obvious way, then $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$.

Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ and let $i_\lambda : M_\lambda \hookrightarrow M$ be the natural monomorphism which embeds M_λ as a submodule of M. Let $\pi_\lambda : M \twoheadrightarrow M_\lambda$ be the epimorphism given by $\pi_\lambda(\Sigma_\mu m_\mu) = m_\lambda$, where $m_\mu \in M_\mu$, all $\mu \in \Lambda$. We have

$$\pi_\lambda i_\lambda = 1 : M_\lambda \to M_\lambda$$

and $\quad \pi_\mu i_\lambda = 0 : M_\lambda \to M_\mu \quad (\lambda \neq \mu)$

where we are writing 1 for the identity isomorphism and 0 for the zero homomorphism (that is, the homomorphism whose image is the zero submodule). It is hoped that it will always be clear from the context when the symbols 0 and 1 are being used to denote maps and when they are being used to denote elements of a ring or module. Note that, since we are writing maps on the left of the elements on which they act, the map $\pi_\lambda i_\lambda$ means the composite map obtained by applying first i_λ and then π_λ. Now let

$$N = \bigcup_{\lambda \in \Lambda} M_\lambda = \bigcup_{\lambda \in \Lambda} i_\lambda(M_\lambda).$$

Clearly N generates M.

Conversely, given R-modules M, M_λ $(\lambda \in \Lambda)$ and homomorphisms $i_\lambda : M_\lambda \to M$, $\pi_\lambda : M \to M_\lambda$ such that $\pi_\lambda i_\lambda = 1$ for each λ and $\pi_\mu i_\lambda = 0$ for $\lambda \neq \mu$, and such that $\bigcup_{\lambda \in \Lambda} i_\lambda(M_\lambda)$ generates M, then $M \simeq \bigoplus_{\lambda \in \Lambda} M_\lambda$. For we can construct a map $\bigoplus_{\lambda \in \Lambda} M_\lambda \to M$ by $(m_\lambda) \mapsto \Sigma_\lambda i_\lambda(m_\lambda)$, and this is clearly an isomorphism.

Given an R-module N and homomorphisms $f_\lambda : M_\lambda \to N$, there is a unique homomorphism $f : \bigoplus_{\lambda \in \Lambda} M_\lambda \to N$ such that $fi_\lambda = f_\lambda$ for all λ: it is given by $f[(m_\lambda)] = \Sigma_\lambda f_\lambda(m_\lambda)$. The last expression makes sense since $m_\lambda = 0$ for almost all λ.

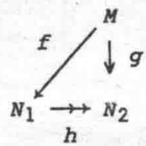
Note that R itself is a left R-module in a natural way; the submodules of R are precisely the left ideals of R. Given an R-module M and $m \in M$ there is a unique homomorphism $f : R \to M$ with $f(1) = m$: it is given by $f(r) = rm$, all $r \in R$.

1.2 *Free modules*

The R-module M is *free* if there is a subfamily $\{m_\lambda\}_{\lambda\in\Lambda}$ of M such that every element $m \in M$ can be written uniquely, except for zeros and the order of the terms, as a finite sum $m = \Sigma_\lambda\, r_\lambda m_\lambda$, where $r_\lambda \in R$, all λ. The set $\{m_\lambda\}_{\lambda\in\Lambda}$ is called a *basis*, or *R-basis*, of M. Given such M, it is clear that if $M_\lambda = R\{m_\lambda\}$ then M_λ is a submodule of M, $M_\lambda \simeq R$, all λ, and $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Conversely, if $M = \oplus_{\lambda\in\Lambda} M_\lambda$, where $M_\lambda \simeq R$ for all λ, then M is free. The proof consists of choosing a suitable basis, and the details are left to the reader.

If M is free with basis $\{m_\lambda\}_{\lambda\in\Lambda}$ and N is an R-module, then any map $f : \{m_\lambda\}_{\lambda\in\Lambda} \to N$ extends to a unique homomorphism $f : M \to N$. For f extends to $M_\lambda = R\{m_\lambda\}$ by $f(rm_\lambda) = rf(m_\lambda)$, all $r \in R$, and the result follows since $M = \oplus_{\lambda\in\Lambda} M_\lambda$.

Suppose M, N_1, N_2 are R-modules, where M is free, and that $g : M \to N_2$ is a homomorphism and $h : N_1 \twoheadrightarrow N_2$ is an epimorphism. We show how to construct a homomorphism $f : M \to N_1$ with $hf = g$, that is, so that the diagram



commutes. (Such f will not in general be unique.) For, if $\{m_\lambda\}_{\lambda\in\Lambda}$ is a basis of M, then $g(m_\lambda) \in N_2$, each $\lambda \in \Lambda$, and since h is surjective we can choose $n_\lambda \in N_1$ with $h(n_\lambda) = g(m_\lambda)$, each $\lambda \in \Lambda$. Since M is free, we can define f by $f(m_\lambda) = n_\lambda$, all λ, and then extend to the whole of M as above; the fact that $hf = g$ follows from the fact that $hf(m_\lambda) = g(n_\lambda)$, all $\lambda \in \Lambda$.

1.3 *Projective modules*

We see now that the above property of free modules is shared by a larger class of modules, called projective modules. The R-module P is *projective* if it satisfies the following equivalent conditions:

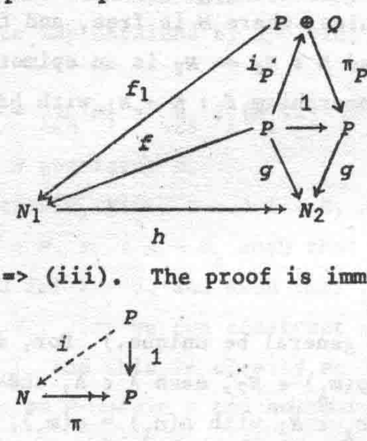
(i) There exists a module Q such that $P \oplus Q$ is free

(ii) Given modules N_1, N_2 and homomorphisms $g : P \to N_2$ and $h : N_1 \twoheadrightarrow N_2$, there is a homomorphism $f : P \to N_1$ with $hf = g$

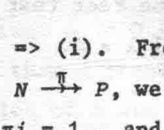
(iii) Given a module N and an epimorphism $\pi : N \twoheadrightarrow P$, then π *splits*, that is, there is a monomorphism $i : P \hookrightarrow N$ with $\pi i = 1 : P \to P$.

To show the equivalence of these conditions, we prove (i) => (ii) => (iii) => (i).

(i) => (ii). By the previous argument, there is a homomorphism $f_1 : P \oplus Q \to N$ with $hf_1 = g\pi_P$, where $\pi_P : P \oplus Q \to P$ is the natural epimorphism. Put $f = f_1 i_P$, where $i_P : P \to P \oplus Q$ is the natural monomorphism. Then $hf = h(f_1 i_P) = (hf_1)i_P = (g\pi_P)i_P = g(\pi_P i_P) = g1_P = g$, as required. The appropriate diagram is:



(ii) => (iii). The proof is immediate:



(The dotted arrow indicates the map whose existence is asserted.)

(iii) => (i). From (iii), if

$N \overset{\pi}{\twoheadrightarrow} P$, we obtain $P \overset{i}{\hookrightarrow} N$

with $\pi i = 1_P$, and it follows that

$N = \ker \pi \oplus i(P) \cong \ker \pi \oplus P$.

For we have the inclusion $\ker \pi \hookrightarrow N$, and the map $N \twoheadrightarrow \ker \pi$ is given by $m \mapsto m - i\pi(m)$, all $m \in N$. Now, given P, choose a generating set $\{m_\lambda\}_{\lambda\in\Lambda}$, and let N be free with basis $\{n_\lambda\}_{\lambda\in\Lambda}$. Then π is defined by extending the map $n_\lambda \overset{\pi}{\longmapsto} m_\lambda$ $(\lambda \in \Lambda)$, and the result follows.

Note: if P, P_1 are projective, so is $P \oplus P_1$.

A module M is *finitely generated* if there is a finite subset N of M with $RN = M$. If we write $R^n = R \oplus R \oplus \ldots \oplus R$ (n terms), then R^n is finitely generated (it is free with a finite basis), and M is finitely generated if and only if there is a natural number n and an epimorphism $R^n \twoheadrightarrow M$.

Note: if M, M_1 are finitely generated, so is $M \oplus M_1$. Further, the module P is finitely generated and projective if and only if there is a module Q and a natural number n such that $P \oplus Q \cong R^n$. Of course Q is then finitely generated and projective also. In particular, R^n is finitely generated and projective.