ALGEBRAIC TOPOLOGY

C. R. F. MAUNDER

ALGEBRAIC TOPOLOGY

C. R. F. MAUNDER

Fellow of Christ's College and University Lecturer
in Pure Mathematics, Cambridge

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE

LONDON NEW YORK NEW ROCHELLE

MELBOURNE SYDNEY

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia

First published by Van Nostrand Reinhold (UK) Ltd 1970
First published by the Cambridge University Press 1980

First printed in Great Britain by Lewis Reprints Ltd, London and Tonbridge
Reprinted in Great Britain at the University Press, Cambridge

British Library cataloguing in publication data

Maunder, Charles Richard Francis
Algebraic topology.
1. Algebraic topology
I. Title
514'.2 QA612 79-41610
ISBN 0 521 23161 2 hard covers
ISBN 0 521 29840 7 paperback

INTRODUCTION

Most of this book is based on lectures to third-year undergraduate and first-year postgraduate students. It aims to provide a thorough grounding in the more elementary parts of algebraic topology, although these are treated wherever possible in an up-to-date way. The reader interested in pursuing the subject further will find suggestions for further reading in the notes at the end of each chapter.

Chapter 1 is a survey of results in algebra and analytic topology that will be assumed known in the rest of the book. The knowledgeable reader is advised to read it, however, since in it a good deal of standard notation is set up. Chapter 2 deals with the topology of simplicial complexes, and Chapter 3 with the fundamental group. The subject of Chapters 4 and 5 is homology and cohomology theory (particularly of simplicial complexes), with applications including the Lefschetz Fixed-Point Theorem and the Poincaré and Alexander duality theorems for triangulable manifolds. Chapters 6 and 7 are concerned with homotopy theory, homotopy groups and CW-complexes, and finally in Chapter 8 we shall consider the homology and cohomology of CW-complexes, giving a proof of the Hurewicz theorem and a treatment of products in cohomology.

A feature of this book is that we have included in Chapter 2 a proof of Zeeman's version of the *relative* Simplicial Approximation Theorem. We believe that the small extra effort needed to prove the relative rather than the absolute version of this theorem is more than repaid by the easy deduction of the equivalence of singular and simplicial homology theory for polyhedra.

Each chapter except the first contains a number of exercises, most of which are concerned with further applications and extensions of the theory. There are also notes at the end of each chapter, which are partly historical and partly suggestions for further reading.

Each chapter is divided into numbered sections, and Definitions, Propositions, Theorems, etc., are numbered consecutively within each section: thus for example Definition 1.2.6 follows Theorem 1.2.5 in the second section (Section 1.2) of Chapter 1. A reference to Exercise n denotes Exercise n at the end of the chapter in which the reference is made; if reference is made to an exercise in a different chapter, then the number of that chapter will also be specified. The symbol ▮ denotes the end (or absence) of a proof, and is also used to indicate the end of an example in the text. References are listed and numbered at the end of the book, and are referred to in the text by numbers in brackets: thus for example [73] denotes the book *Homotopy Theory* by S.-T. Hu.

Finally, it is a pleasure to acknowledge the help I have received in writing this book. My indebtedness to the books of Seifert and Threlfall [124] and Hu [73], and papers by Puppe [119], G. W. Whitehead [155], J. H. C. Whitehead [160] and Zeeman [169] will be obvious to anyone who has read them, but I should also like to thank D. Barden, R. Brown, W. B. R. Lickorish, N. Martin, R. Sibson, A. G. Tristram and the referee for many valuable conversations and suggestions.

CONTENTS

CHAPTER 1

ALGEBRAIC AND TOPOLOGICAL PRELIMINARIES

1.1 Introduction

In this chapter we collect together some elementary results in set theory, algebra and analytic topology that will be assumed known in the rest of the book. Since the reader will probably be familiar with most of these results, we shall usually omit proofs and give only definitions and statements of theorems. Proofs of results in set theory and analytic topology will be found in Kelley [85], and in algebra in Jacobson [77]; or indeed in almost any other standard textbook. It will be implicitly assumed that the reader is familiar with the concepts of sets (and subsets), integers, and rational, real and complex numbers.

1.2 Set theory

The notation $a \in A$ means that a is an element of the set A; $A \subset B$ that A is a subset of B. $\{a \in A \mid \ldots\}$ means the subset of A such that $\ldots$ is true, and if A, B are subsets of some set C, then $A \cup B$, $A \cap B$ denote the *union* and *intersection* of A and B respectively: thus $A \cup B = \{c \in C \mid c \in A \text{ or } c \in B\}$ and $A \cap B = \{c \in C \mid c \in A \text{ and } c \in B\}$. Unions and intersections of arbitrary collections of sets are similarly defined.

Definition 1.2.1 Given sets A and B, the *product set* $A \times B$ is the set of all ordered pairs (a, b), for all $a \in A$, $b \in B$. A *relation* between the sets A and B is a subset R of $A \times B$; we usually write aRb for the statement '$(a, b) \in R$'.

Definition 1.2.2 A *partial ordering* on a set A is a relation $<$ between A and itself such that, whenever $a < b$ and $b < c$, then $a < c$. A *total ordering* on A is a partial ordering $<$ such that

(a) if $a < b$ and $b < a$, then $a = b$;
(b) given $a, b \in A$, either $a < b$ or $b < a$.

Proposition 1.2.3 *Given a finite set A containing n distinct elements, there exist $n!$ distinct total orderings on A.* ▮

Definition 1.2.4 A relation R between a set A and itself is called an *equivalence relation on* A if

(a) for all $a \in A$, aRa;
(b) if aRb, then bRa;
(c) if aRb and bRc, then aRc.

The *equivalence class* $[a]$ of an element $a \in A$ is defined by $[a] = \{b \in A \mid aRb\}$.

Theorem 1.2.5 *If R is an equivalence relation on A, then each element of A is in one and only one equivalence class.* ∎

Definition 1.2.6 Given sets A and B, a *function* f from A to B is a relation between A and B such that, for each $a \in A$, there exists a unique $b \in B$ such that afb. We write $b = f(a)$, or $f(a) = b$, for the statement 'afb', and $f: A \to B$ for 'f is a function from A to B'.

Example 1.2.7 Given any set A, the *identity function* $1_A: A \to A$ is defined by $1_A(a) = a$ for all $a \in A$ (we shall often abbreviate 1_A to 1, if no ambiguity arises). ∎

Definition 1.2.8 If $f: A \to B$ is a function and C is a subset of A, the *restriction* $(f|C): C \to B$ is defined by $(f|C)(c) = f(c)$ for all $c \in C$. Given two functions $f: A \to B$, $g: B \to C$, the *composite function* $gf: A \to C$ is defined by $gf(a) = g(f(a))$. The *image* $f(A)$ of $f: A \to B$ is the subset of B of elements of the form $f(a)$, for some $a \in A$; f is *onto* if $f(A) = B$; f is *one-to-one* (written (1-1) if, whenever $f(a_1) = f(a_2)$, then $a_1 = a_2$; f is a *(1-1)-correspondence* if it is both onto and (1-1). Two sets A and B are said to be *in (1-1)-correspondence* if there exists a (1-1)-correspondence $f: A \to B$.

Proposition 1.2.9 *Let $f: A \to B$ be a function.*

(a) *$f: A \to B$ is onto if and only if there exists a function $g: B \to A$ such that $fg = 1_B$.*

(b) *$f: A \to B$ is (1-1) if and only if there exists a function $g: B \to A$ such that $gf = 1_A$ (provided A is non-empty).*

(c) *$f: A \to B$ is a (1-1)-correspondence if and only if there exists a function $g: B \to A$ such that $fg = 1_B$ and $gf = 1_A$. In this case g is unique and is called the 'inverse function' to f.* ∎

Definition 1.2.10 A set A is *countable* (or *enumerable*) if it is in (1-1)-correspondence with a subset of the set of positive integers.

Proposition 1.2.11 *If the sets A and B are countable, so is $A \times B$.* ∎

Definition 1.2.12 A *permutation* of a set A is a (1-1)-correspondence from A to itself; a *transposition* is a permutation that leaves fixed all but two elements of A, which are interchanged. If A is a finite set, a permutation is *even* if it is a composite of an even number of transpositions and *odd* if it is a composite of an odd number of transpositions.

1.3 Algebra

Definition 1.3.1 A *group* G is a set, together with a function $m: G \times G \to G$, called a *multiplication*, satisfying the following rules.

(a) $m(m(g_1, g_2), g_3) = m(g_1, m(g_2, g_3))$ for all $g_1, g_2, g_3 \in G$.

(b) There exists an element $e \in G$, called the *unit element*, such that $m(g, e) = g = m(e, g)$ for all $g \in G$.

(c) For each $g \in G$, there exists $g' \in G$ such that $m(g, g') = e = m(g', g)$.

The element $m(g_1, g_2)$ is regarded as the 'product' of g_1 and g_2, and is normally written g_1g_2, so that rule (a), for example, becomes $(g_1g_2)g_3 = g_1(g_2g_3)$ (this is usually expressed by saying that the product is *associative*; we may unambiguously write $g_1g_2g_3$ for either $(g_1g_2)g_3$ or $g_1(g_2g_3)$). We shall often write 1 instead of e in rule (b), and g^{-1} instead of g' in rule (c) (g^{-1} is the *inverse* of g).

The *order* of G is the number of elements in it, if this is finite; the *order* of the element $g \in G$ is the smallest positive integer n such that $g^n = e$ (where g^n means the product of g with itself n times).

A group with just one element is called a *trivial group*, often written 0.

A subset H of a group G is called a *subgroup* if $m(H \times H) \subset H$ and H satisfies rules (a)–(c) with respect to m.

Proposition 1.3.2 *A non-empty subset H of G is a subgroup if and only if $g_1g_2^{-1} \in H$ for all $g_1, g_2 \in H$.* ∎

Theorem 1.3.3 *If H is a subgroup of a finite group G, the order of H divides the order of G.* ∎

Definition 1.3.4 Given groups G and H, a *homomorphism* $\theta: G \to H$ is a function such that $\theta(g_1g_2) = \theta(g_1)\theta(g_2)$ for all $g_1, g_2 \in G$. θ is an *isomorphism* (or is *isomorphic*) if it is also a (1-1)-correspondence; in this case G and H are said to be *isomorphic*, written $G \cong H$. We write Im θ for $\theta(G)$, and the *kernel* of θ, Ker θ, is the subset $\{g \in G \mid \theta(g) = e\}$, where e is the unit element of H.

Example 1.3.5 The identity function $1_G\colon G \to G$ is an isomorphism, usually called the *identity isomorphism.* ▮

Proposition 1.3.6

(a) *The composite of two homomorphisms is a homomorphism.*

(b) *If θ is an isomorphism, the inverse function is also an isomorphism.*

(c) *If $\theta\colon G \to G$ is a homomorphism,* Im θ *is a subgroup of H and* Ker θ *is a subgroup of G. θ is* (1-1) *if and only if* Ker θ *contains only the unit element of G.* ▮

Definition 1.3.7 Two elements $g_1, g_2 \in G$ are *conjugate* if there exists $h \in G$ such that $g_2 = h^{-1}g_1h$. A subgroup H of G is *normal* (*self-conjugate*) if $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$.

Given a normal subgroup H of a group G, define an equivalence relation R on G by the rule g_1Rg_2 if and only if $g_1g_2^{-1} \in H$; then R is an equivalence relation and the equivalence class $[g]$ is called the *coset* of g.

Theorem 1.3.8 *The set of distinct cosets can be made into a group by setting* $[g_1][g_2] = [g_1g_2]$. ▮

Definition 1.3.9 The group of Theorem 1.3.8 is called the *quotient group* of G by H, and is written G/H.

Proposition 1.3.10 *The function $p\colon G \to G/H$, defined by $p(g) = [g]$, is a homomorphism, and is onto.* Ker $p = H$. ▮

Theorem 1.3.11 *Given groups G, G', normal subgroups H, H' of G, G' respectively, and a homomorphism $\theta\colon G \to G'$ such that $\theta(H) \subset H'$, there exists a unique homomorphism $\bar{\theta}\colon G/H \to G'/H'$ such that* $\bar{\theta}[g] = [\theta(g)]$. ▮

Proposition 1.3.12 *Given a homomorphism $\theta\colon G \to H$,* Ker p *is a normal subgroup of G, and $\bar{\theta}\colon G/$*Ker $\theta \to$ Im θ *is an isomorphism.* ▮

Definition 1.3.13 Given a collection of groups G_a, one for each element a of a set A (not necessarily finite), the *direct sum* $\bigoplus_{a \in A} G_a$ is the set of collections of elements (g_a), one element g_a in each G_a, where all but a finite number of the g_a's are unit elements. The multiplication in $\bigoplus_{a \in A} G_a$ is defined by $(g_a)(g'_a) = (g_ag'_a)$, that is, corresponding elements in each G_a are multiplied together.

We shall sometimes write $\oplus G_a$ instead of $\bigoplus_{a \in A} G_a$, if no ambiguity can arise; and if A is the set of positive integers we write $\bigoplus_{n=1}^{\infty} G_n$ (similarly $\bigoplus_{r=1}^{n} G_r$ or even $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ if A is the set of the first n positive integers). In the latter case, we prefer the notation $g_1 \oplus g_2 \oplus \cdots \oplus g_n$ rather than (g_r) for a typical element.

Proposition 1.3.14 *Given homomorphisms* $\theta_a\colon G_a \to H_a$ $(a \in A)$, *the function* $\oplus \theta_a\colon \bigoplus_{a \in A} G_a \to \bigoplus_{a \in A} H_a$, *defined by* $\oplus \theta_a(g_a) = (\theta_a(g_a))$, *is a homomorphism, which is isomorphic if each* θ_a *is.* ▮

Once again, we prefer the notation $\theta_1 \oplus \theta_2 \oplus \cdots \oplus \theta_n$ if A is the set of the first n integers.

Definition 1.3.15 Given a set A, the *free group generated by* A, Gp $\{A\}$, is defined as follows. A *word* w in A is a formal expression

$$w = a_1^{\epsilon_1} \cdots a_n^{\epsilon_n},$$

where $a_1, \ldots, a_n$ are (not necessarily distinct) elements of A, $\epsilon_i = \pm 1$, and $n \geqslant 0$ (if $n = 0$, w is the 'empty word', and is denoted by 1). Define an equivalence relation R on the set of words in A by the rule: $w_1 R w_2$ if and only if w_2 can be obtained from w_1 by a finite sequence of operations of the form 'replace $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ by $a_1^{\epsilon_1} \cdots a_r^{\epsilon_r} a^1 a^{-1} a_{r+1}^{\epsilon_{r+1}} \cdots a_n^{\epsilon_n}$ or $a^{\epsilon_1} \cdots a_r^{\epsilon_r} a^{-1} a^1 a_{r+1}^{\epsilon_{r+1}} \cdots a_n^{\epsilon_n}$ $(0 \leqslant r \leqslant n)$, or vice versa'. The elements of Gp $\{A\}$ are the equivalence classes $[w]$ of words in A, and the multiplication is defined by

$$[a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}][a_{n+1}^{\epsilon_{n+1}} \cdots a_m^{\epsilon_m}] = [a_1^{\epsilon_1} \cdots a_n^{\epsilon_n} a_{n+1}^{\epsilon_{n+1}} \cdots a_m^{\epsilon_m}].$$

Normally the elements of Gp $\{A\}$ are written without square brackets, and by convention we write a for a^1, a^2 for $a^1 a^1$, a^{-2} for $a^{-1} a^{-1}$, and so on. The omission of square brackets has the effect of introducing equalities such as $a^2 a^{-1} = a$, $aa^{-1} = 1$ (note that 1 is the unit element of Gp $\{A\}$).

Example 1.3.16 The group of integers under addition (usually denoted by Z) is isomorphic to Gp $\{a\}$, where a denotes a set consisting of just one element a. ▮

Proposition 1.3.17 *Given a set* A, *a group* G *and a function* $\theta\colon A \to G$, *there exists a unique homomorphism* $\bar{\theta}\colon \mathrm{Gp}\,\{A\} \to G$ *such that* $\bar{\theta}(a) = \theta(a)$ *for each* $a \in A$. ▮

Definition 1.3.18 Given a set B of elements of Gp $\{A\}$, let $\bar{B}$ be the intersection of all the normal subgroups of Gp $\{A\}$ that contain B. $\bar{B}$ is itself a normal subgroup (called the *subgroup generated by* B), and the quotient group Gp $\{A\}/\bar{B}$ is called the group *generated by* A, *subject to the relations* B, and is written Gp $\{A; B\}$. The elements of Gp $\{A; B\}$ are still written in the form of words in A, and the effect of the relations B is to introduce new equalities of the form $b = 1$, for each element $b \in B$.

A group G is *finitely generated* if $G \cong$ Gp $\{A; B\}$ for some finite set A; in particular, if A has only one element, G is said to be *cyclic*.

Example 1.3.19 For each integer $n \geqslant 2$, the group Z_n of integers modulo n, under addition mod n, is a cyclic group, since $Z_n \cong$ Gp $\{a; a^n\}$.

In fact every group G is isomorphic to a group of the form Gp $\{A; B\}$, since we could take A to be the set of all the elements of G. Of course, this representation is not in general unique: for example, Gp $\{a; a^2\} \cong$ Gp $\{a, b; a^2, b\}$.

Proposition 1.3.20 *A function $\theta: A \rightarrow G$, such that $\theta(b) = e$ (the unit element of G) for all $b \in B$, defines a unique homomorphism $\bar{\theta}$:* Gp $\{A; B\} \rightarrow G$, *such that $\bar{\theta}(a) = \theta(a)$ for all $a \in A$.* ∎

Definition 1.3.21 A group G is said to be *abelian* (*commutative*) if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$. In an abelian group, the notation $g_1 + g_2$ is normally used instead of $g_1 g_2$ (and the unit element is usually written 0). Similarly, one writes $-g$ instead of g^{-1}.

Observe that every subgroup of an abelian group is normal, and that every quotient group of an abelian group is abelian, as also is every direct sum of a collection of abelian groups.

Definition 1.3.22 Given a group G (not necessarily abelian), the *commutator subgroup* $[G, G]$ is the set of all (finite) products of elements of the form $g_1 g_2 g_1^{-1} g_2^{-1}$.

Proposition 1.3.23 *$[G, G]$ is a normal subgroup of G, and $G/[G, G]$ is abelian. Given any homomorphism $\theta: G \rightarrow H$ into an abelian group,* $[G, G] \subset \operatorname{Ker} \theta$. ∎

Proposition 1.3.24 *If $G \cong H$, then $G/[G, G] \cong H/[H, H]$.* ∎

Definition 1.3.25 Given a set A, the *free abelian group generated by* A, Ab $\{A\}$, is the group Gp $\{A\}/[$Gp $\{A\}$, Gp $\{A\}]$.

Proposition 1.3.26 Ab $\{A\} \cong$ Gp $\{A; B\}$, *where B is the set of all elements of* Gp $\{A\}$ *of the form* $a_1 a_2 a_1^{-1} a_2^{-1}$. ▮

The elements of Ab $\{A\}$ will normally be written in the form $\epsilon_1 a_1 + \cdots + \epsilon_n a_n$ ($\epsilon_i = \pm 1$), and the coset of 1 will be denoted by 0.

Definition 1.3.27 If B is a set of elements of Ab $\{A\}$, let $\bar{B}$ be the intersection of all the subgroups of Ab $\{A\}$ that contain B: thus $\bar{B}$ is a subgroup and consists of all finite sums of elements of B (or their negatives), together with 0. The quotient group Ab $\{A\}/\bar{B}$ is called the *abelian group generated by A, subject to the relations B*, and is written Ab $\{A; B\}$.

As in Definition 1.3.18, the elements of Ab $\{A; B\}$ are still written in the form of 'additive' words in A.

Proposition 1.3.28 *If $G =$ Gp $\{A; B\}$, and $p: G \to G/[G, G]$ is the homomorphism of Proposition 1.3.10, then $G/[G, G] \cong$ Ab$\{A; p(B)\}$.* ▮

Examples 1.3.29 Particular examples of abelian groups include Z and Z_n: observe that $Z \cong$ Ab $\{a\}$ and $Z_n \cong$ Ab $\{a; na\}$. We shall also make frequent use of the groups of rational, real and complex numbers, under addition: these are denoted by R, Q and C respectively. ▮

There is a very useful theorem giving a standard form for the finitely generated abelian groups.

Theorem 1.3.30 *Let G be a finitely generated abelian group. There exists an integer $n \geqslant 0$, primes $p_1, \ldots, p_m$ and integers $r_1, \ldots, r_m$ ($m \geqslant 0$, $r_i \geqslant 1$), such that*

$$G \cong nZ \oplus Z_{p_1^{r_1}} \oplus \cdots \oplus Z_{p_m^{r_m}}.$$

(Here, nZ denotes the direct sum of n copies of Z.) Moreover, if

$$H \cong lZ \oplus Z_{q_1^{s_1}} \oplus \cdots \oplus Z_{q_k^{s_k}},$$

then $G \cong H$ if and only if $n = l$, $m = k$, and the numbers $p_1^{r_1}, \ldots, p_m^{r_m}$ and $q_1^{s_1}, \ldots, q_k^{s_k}$ are equal in pairs. ▮

Definition 1.3.31 A sequence of groups and homomorphisms

$$\cdots \longrightarrow G_i \xrightarrow{\theta_i} G_{i+1} \xrightarrow{\theta_{i+1}} G_{i+2} \longrightarrow \cdots$$

is called an *exact sequence* if, for each i, Ker $\theta_i =$ Im θ_{i-1} (if the sequence terminates in either direction, for example $G_0 \xrightarrow{\theta_0} G_1 \to \cdots$

or $\cdots \longrightarrow G_{n-1} \xrightarrow{\theta_{n-1}} G_n$, then no restriction is placed on Ker θ_0 or Im θ_{n-1}).

Example 1.3.32 The sequence $0 \to G \xrightarrow{\theta} H \to 0$ is exact if and only if θ is an isomorphism. (Here, 0 denotes the trivial group, and $0 \to G$, $H \to 0$ the only possible homomorphisms.) This follows immediately from the definitions.

Similarly, if H is a normal subgroup of G and $i\colon H \to G$ is defined by $i(h) = h$ for all $h \in H$, then

$$0 \to H \xrightarrow{i} G \xrightarrow{p} G/H \to 0$$

is an exact sequence. ∎

Proposition 1.3.33 *Given exact sequences*

$$0 \to G_a \xrightarrow{\theta_a} H_a \xrightarrow{\phi_a} K_a \to 0,$$

one for each element a of a set A, the sequence

$$0 \longrightarrow \bigoplus_{a\in A} G_a \xrightarrow{\oplus\theta_a} \bigoplus_{a\in A} H_a \xrightarrow{\oplus\phi_a} \bigoplus_{a\in A} K_a \longrightarrow 0$$

is also exact. ∎

Definition 1.3.34 A square of groups and homomorphisms

$$\begin{array}{ccc} G_1 & \xrightarrow{\theta_1} & G_2 \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_2} \\ H_1 & \xrightarrow[\theta_2]{} & H_2 \end{array}$$

is said to be *commutative* if $\phi_2\theta_1 = \theta_2\phi_1$. Commutative triangles, etc., are similarly defined, and in general any diagram of groups and homomorphisms is *commutative* if each triangle, square, ... in it is commutative.

Proposition 1.3.35 *Given a commutative diagram of groups and homomorphisms*

$$\begin{array}{ccccccccc} G_1 & \xrightarrow{\theta_1} & G_2 & \xrightarrow{\theta_2} & G_3 & \xrightarrow{\theta_3} & G_4 & \xrightarrow{\theta_4} & G_5 \\ {\scriptstyle\psi_1}\downarrow & & {\scriptstyle\psi_2}\downarrow & & {\scriptstyle\psi_3}\downarrow & & {\scriptstyle\psi_4}\downarrow & & {\scriptstyle\psi_5}\downarrow \\ H_1 & \xrightarrow[\phi_1]{} & H_2 & \xrightarrow[\phi_2]{} & H_3 & \xrightarrow[\phi_3]{} & H_4 & \xrightarrow[\phi_4]{} & H_5, \end{array}$$

in which the rows are exact sequences, and ψ_2, ψ_4 are isomorphisms, ψ_1 is onto and ψ_5 is (1-1), then ψ_3 is an isomorphism.

Proof. To show that ψ_3 is (1-1), consider an element $x \in G_3$ such that $\psi_3(x) = 1$ (we shall write 1 indiscriminately for the unit element of each group). Then $\psi_4\theta_3(x) = \phi_3\psi_3(x) = 1$, so that $\theta_3(x) = 1$ since ψ_4 is isomorphic. By exactness, therefore, $x = \theta_2(y)$ for some $y \in G_2$; and then $\phi_2\psi_2(y) = \psi_3\theta_2(y) = 1$. By exactness again, $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$; and $z = \psi_1(w)$ for some $w \in G_1$ since ψ_1 is onto. Thus $\psi_2\theta_1(w) = \phi_1\psi_1(w) = \psi_2(y)$, so that $\theta_1(w) = y$; but then $x = \theta_2(y) = \theta_2\theta_1(w) = 1$.

The proof that ψ_3 is onto is rather similar. This time, choose an element $x \in H_3$; then $\phi_3(x) = \psi_4(y)$ for some $y \in G_4$, since ψ_4 is isomorphic. Thus $\psi_5\theta_4(y) = \phi_4\psi_4(y) = \phi_4\phi_3(x) = 1$, so that $\theta_4(y) = 1$ since ψ_5 is (1-1). Hence by exactness $y = \theta_3(z)$ for some $z \in G_3$. Unfortunately there is no reason why $\psi_3(z)$ should be x, but it is at least true that $\phi_3((\psi_3(z))^{-1}x) = (\psi_4\theta_3(z))^{-1}(\phi_3(x)) = 1$, so that $(\psi_3(z))^{-1}x = \phi_2\psi_2(w)$ for some $w \in G_2$, since ψ_2 is isomorphic. Thus $\psi_3(z \,.\, \theta_2(w)) = (\psi_3(z)).\phi_2\psi_2(w) = (\psi_3(z))(\psi_3(z))^{-1}x = x$, and hence ψ_3 is onto. ∎

Proposition 1.3.36 *Given an exact sequence of abelian groups and homomorphisms*

$$0 \to G \xrightarrow{\theta} H \xrightarrow{\phi} K \to 0,$$

and a homomorphism $\psi\colon K \to H$ such that $\phi\psi = 1_K$, then $H \cong G \oplus K$.

Proof. Define $\alpha\colon G \oplus K \to H$ by $\alpha(g \oplus k) = \theta(g) + \psi(k)$: it is easy to see that α is a homomorphism. Also α is (1-1), for if $\alpha(g \oplus k) = 0$, we have

$$0 = \phi(\theta(g) + \psi(k)) = \phi\psi(k) = k;$$

but then $\theta(g) = 0$, so that $g = 0$ since θ is (1-1).

Moreover α is onto, since given $h \in H$ we have

$$\phi(h - \psi\phi(h)) = \phi(h) - \phi\psi\phi(h) = 0.$$

Thus there exists $g \in G$ such that $h - \psi\phi(h) = \theta(g)$, that is,

$$h = \theta(g) + \psi\phi(h) = \alpha(g \oplus \phi(h)). \; ∎$$

An exact sequence as in the statement of Proposition 1.3.36 is called a *split exact sequence*.

Of course, it is not true that all exact sequences $0 \to G \to H \to K \to 0$ split. However, this is true if K is a *free* abelian group.

Proposition 1.3.37 *Given abelian groups and homomorphisms $G \xrightarrow{\theta} H \xleftarrow{\phi} K$, where θ is onto and K is free abelian, there exists a homomorphism $\psi\colon K \to G$ such that $\theta\psi = \phi$.*