London Mathematical Society Lecture Note Series 431

Harmonic and Subharmonic Function Theory on the Hyperbolic Ball

Manfred Stoll

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London Mathematical Society Lecture Note Series: 431

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University Printing House, Cambridge CB2 8BS, United Kingdom

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www.cambridge.org
Information on this title: www.cambridge.org/9781107541481

First published 2016

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloguing in Publication Data
Names: Stoll, Manfred.
Title: Harmonic and subharmonic function theory on the hyperbolic ball / Manfred Stoll, University of South Carolina.
Description: Cambridge : Cambridge University Press, 2016. | Series: London Mathematical Society lecture note series ; 431 | Includes bibliographical references and index.
Identifiers: LCCN 2015049530 | ISBN 9781107541481 (pbk.)
Subjects: LCSH: Harmonic functions. | Subharmonic functions. | Hyperbolic spaces.
Classification: LCC QA405 .S76 2016 | DDC 515/.53–dc23
LC record available at http://lccn.loc.gov/2015049530

ISBN 978-1-107-54148-1 Paperback

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To Mary Lee

Preface

The intent of these notes is to provide a detailed and comprehensive treatment of harmonic and subharmonic function theory on hyperbolic space in $\mathbb{R}^n$. Although our primary emphasis will be in the setting of the unit ball $\mathbb{B}$ with hyperbolic metric ds given by

$$ds = \frac{2|dx|}{1 - |x|^2}, \qquad (1)$$

we will also consider the analogue of many of the results in the hyperbolic half-space $\mathbb{H}$. Undoubtedly some of the results are known, either in the setting of rank one noncompact symmetric spaces (e.g. [38]), or more generally, in Riemannian spaces (e.g. [13]). An excellent introduction to harmonic function theory on noncompact symmetric spaces can be found in the survey article [47] by A. Koranyi. The 1973 paper by K. Minemura [57] provides an introduction to harmonic function theory on real hyperbolic space considered as a rank one noncompact symmetric space. Other contributions to the subject area in this setting will be indicated in the text.

With the goal of making these notes accessible to a broad audience, our approach does not require any knowledge of Lie groups and only a limited knowledge of differential geometry. The development of the theory is analogous to the approach taken by W. Rudin [72] and by the author [84] in their development of Möbius invariant harmonic function theory on the hermitian ball in $\mathbb{C}^n$. Although our primary emphasis is on harmonic function theory on the ball, we do include many relevant results for the hyperbolic upper half-space $\mathbb{H}$, both in the text and in the exercises. With only one or two exceptions, the notes are self-contained with the only prerequisites being a standard beginning graduate course in real analysis.

In Chapter 1 we provide a brief review of Möbius transformation in $\mathbb{R}^n$. This is followed in Chapter 2 by a characterization of the group $\mathcal{M}(\mathbb{B})$ of

Möbius self-maps of the unit ball $\mathbb{B}$ in $\mathbb{R}^n$. As in [72] we define a family $\{\varphi_a : a \in \mathbb{B}\}$ of Möbius transformations of $\mathbb{B}$ satisfying $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a(\varphi_a(x)) = x$ for all $x \in \mathbb{B}$. Furthermore, for every $\psi \in \mathcal{M}(\mathbb{B})$, it is proved that there exists $a \in \mathbb{B}$ and an orthogonal transformation A such that $\psi = A\varphi_a$. When $n = 2$, the mappings φ_a correspond to the usual analytic Möbius transformations of the unit disc $\mathbb{D}$ given by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}. \tag{2}$$

Some of the properties of the mappings $\{\varphi_a\}$ and of functions in $\mathcal{M}(\mathbb{B})$ are developed in Section 2.1. In this chapter we also introduce the hyperbolic metric in $\mathbb{B}$ and in the hyperbolic half-space $\mathbb{H}$. Most of the results of these two sections are contained in the works of L. V. Ahlfors [4], [5] , and the text by A. F. Beardon [11].

In Chapter 3 we derive the Laplacian, gradient, and measure on $\mathbb{B}$ that are invariant under $\mathcal{M}(\mathbb{B})$. Even though the formula for the Laplacian can be derived from the hyperbolic metric, we will follow the approach of W. Rudin [72, Chapter 4]. For $f \in C^2(\mathbb{B})$ we define $\Delta_h f$ by

$$\Delta_h f(a) = \Delta(f \circ \varphi_a)(0),$$

where Δ is the usual Laplacian in $\mathbb{R}^n$. The operator Δ_h is shown to satisfy $\Delta_h(f \circ \psi)(x) = (\Delta_h f)(\psi(x))$ for all $\psi \in \mathcal{M}(\mathbb{B})$. Furthermore, an explicit computation gives

$$\Delta_h f(x) = (1 - |x|^2)^2 \Delta f(x) + 2(n - 2)(1 - |x|^2)\langle x, \nabla f(x)\rangle,$$

where ∇f is the Euclidean gradient of the function f. In this chapter it is also proved that the Green's function for Δ_h is given by $G_h(x, y) = g(|\varphi_x(y)|)$, where g is the radial function on $\mathbb{B}$ defined by

$$g(r) = \frac{1}{n}\int_r^1 \frac{(1 - s^2)^{n-2}}{s^{n-1}}\, ds.$$

In Theorem 3.3.1 we prove that for $\psi \in \mathcal{M}(\mathbb{B})$, the Jacobian J_ψ of the mapping ψ satisfies

$$|J_\psi(x)| = \frac{(1 - |\psi(x)|^2)^n}{(1 - |x|^2)^n}.$$

From this it now follows that the Möbius invariant measure τ on $\mathbb{B}$ is given by

$$d\tau(x) = (1 - |x|^2)^{-n} d\nu(x),$$

where ν is the normalized volume measure on $\mathbb{B}$. In the exercises we develop the invariant Laplacian, Green's function, and invariant measure on $\mathbb{H}$.

A real-valued C^2 function f on $\mathbb{B}$ is defined to be either $\mathcal{H}$-harmonic or $\mathcal{H}$-subharmonic on $\mathbb{B}$ depending on whether $\Delta_h f = 0$ or $\Delta_h f \geq 0$. It is well known that a continuous function f is harmonic in the unit disc $\mathbb{D}$ if and only if for all r, $0 < r < 1$, and $w \in \mathbb{D}$,

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi_w(re^{it}))\, dt, \tag{3}$$

where φ_w is the Möbius transformation of $\mathbb{D}$ given by (2). The above is called the **invariant mean-value property**. One of the first results proved in Chapter 4 is the following analogue of the invariant mean-value property: A real-valued C^2 function f is $\mathcal{H}$-subharmonic on $\mathbb{B}$ if and only if for all $a \in \mathbb{B}$ and $0 < r < 1$,

$$f(a) \leq \int_{\mathbb{S}} f(\varphi_a(rt)) d\sigma(t), \tag{4}$$

with equality if and only if f is $\mathcal{H}$-harmonic on $\mathbb{B}$. In the above, $\mathbb{S}$ is the unit sphere in $\mathbb{R}^n$, σ is normalized surface measure on $\mathbb{S}$, and φ_a is the Möbius transformation of $\mathbb{B}$ mapping 0 to a with $\varphi_a(\varphi_a(x)) = x$. The integral in (4) is an average of f over the hyperbolic or non-Euclidean sphere $\{\varphi_a(rt) : t \in \mathbb{S}\}$ whose hyperbolic center is a. Inequality (4) is then used in Section 4.3 to extend the definition of $\mathcal{H}$-subharmonic to the class of upper semicontinuous functions on $\mathbb{B}$. The remainder of the chapter is devoted to extending some of the standard results about subharmonic functions to $\mathcal{H}$-subharmonic functions on $\mathbb{B}$. We conclude the chapter with a discussion of quasi-nearly $\mathcal{H}$-subharmonic functions and prove several inequalities involving these functions that will prove useful later in the text.

The Poisson kernel P_h for Δ_h is introduced in Chapter 5. In Section 5.1 we prove using Green's formula that for $(a, t) \in \mathbb{B} \times \mathbb{S}$,

$$P_h(a, t) = -\lim_{r \to 1} nr^{n-1}(1 - r^2)^{2-n} \langle \nabla G_a(rt), t \rangle,$$

where $G_a(rt) = G_h(a, rt)$ is the Green's function for Δ_h. This immediately gives

$$P_h(x, t) = \left(\frac{1 - |x|^2}{|x - t|^2} \right)^{n-1}, \quad (x, t) \in \mathbb{B} \times \mathbb{S}.$$

The standard results for Poisson integrals of continuous functions are included in Section 5.3, and in Section 5.2 we prove a result of P. Jaming [43] that provides an integral representation of the Euclidean Poisson kernel in terms of the hyperbolic Poisson kernel. In Section 5.5 we investigate the eigenfunctions of Δ_h. We close the section with a brief discussion of the Poisson kernel on $\mathbb{H}$.

In Chapter 6 we consider the spherical harmonic expansions of $\mathcal{H}$-harmonic functions. One of the key results of this section is that if p_α is a spherical harmonic of degree α on $\mathbb{S}$, then the Poisson integral $P_h[p_\alpha]$ of p_α is given by

$$P_h[p_\alpha](x) = |x|^\alpha S_{n,\alpha}(|x|)p_\alpha\left(\tfrac{x}{|x|}\right),$$

where $S_{n,\alpha}$ is given by a hypergeometric function. Interestingly, when n is even, $S_{n,\alpha}(r)$ is simply a polynomial in r of degree $n-2$. These results are then used to show how the Poisson integral $P_h[q]$ can be computed for any polynomial q on $\mathbb{S}$. As an example, in $\mathbb{R}^4$, the $\mathcal{H}$-harmonic function with boundary values t_1^2 is given by $P_h[t_1^2](x) = \frac{1}{4} + (2-|x|^2)(x_1^2 - \frac{1}{4}|x|^2)$. In contrast, the Euclidean harmonic function h with boundary values t_1^2 is given by $h(x) = \frac{1}{4}(1-|x|^2) + x_1^2$. Finally, in Section 6.3 we follow the methods of P. Ahern, J. Bruna, and C. Cascante [2] to derive the spherical harmonic expansion of $\mathcal{H}$-harmonic functions on $\mathbb{B}$.

Chapter 7 is devoted to the study of Hardy and Hardy–Orlicz type spaces of $\mathcal{H}$-harmonic and $\mathcal{H}$-subharmonic functions on $\mathbb{B}$. In Chapter 8, we study the boundary behavior of Poisson integrals on $\mathbb{B}$. This chapter contains many of the standard results concerning non-tangential and radial maximal functions. In addition to proving the usual Fatou theorem (Theorem 8.3.3) concerning non-tangential limits of Poisson integrals of measures, we also include a proof of a local Fatou theorem of I. Privalov [68] for $\mathcal{H}$-harmonic functions on $\mathbb{B}$.

The Riesz decomposition theorem for $\mathcal{H}$-subharmonic functions is proved in Chapter 9. The main result of this chapter (Corollary 9.1.3) proves that if f is $\mathcal{H}$-subharmonic on $\mathbb{B}$ and f has an $\mathcal{H}$-harmonic majorant, then

$$f(x) = F_f(x) - \int_{\mathbb{B}} G_h(x,y)\,d\mu_f(y),$$

where μ_f is the Riesz measure of f and F_f is the least $\mathcal{H}$-harmonic majorant of f. In Section 9.2 we include several applications of the Riesz decomposition theorem, including a Hardy–Stein identity for non-negative $\mathcal{H}$-subharmonic functions for which f^p, $p \geq 1$, has an $\mathcal{H}$-harmonic majorant on $\mathbb{B}$. In Section 9.3 we extend a result of D. H. Armitage [8] concerning the integrability of non-negative superharmonic functions. We conclude the chapter by proving that invariant Green potentials of measures have radial limit zero almost everywhere on $\mathbb{S}$, and provide an example of a measure μ for which the Green potential of μ has non-tangential limit $+\infty$ almost everywhere on $\mathbb{S}$.

Finally, in Chapter 10 we introduce and investigate basic properties of weighted Bergman- and Dirichlet-type spaces of $\mathcal{H}$-harmonic functions on $\mathbb{B}$, denoted respectively by $\mathcal{B}_\gamma^p$ and $\mathcal{D}_\gamma^p$. These spaces consist of the set of $\mathcal{H}$-harmonic functions f on $\mathbb{B}$ for which f, respectively $|\nabla^h f|$, are in

$L^p((1-|x|^2)^\gamma d\tau(x))$, $0 < p < \infty$, $\gamma > 0$, where τ is the invariant measure on $\mathbb{B}$ and ∇^h is the invariant gradient on $\mathbb{B}$. One of the main results of this chapter is that if $\gamma > (n-1)$, then $f \in \mathcal{B}^p_\gamma$ if and only if $f \in \mathcal{D}^p_\gamma$ for all p, $0 < p < \infty$. In Section 10.4 we investigate the integrability of functions in $\mathcal{B}^p_\gamma$ and $\mathcal{D}^p_\gamma$ for $\gamma \le (n-1)$. This chapter also contains a discussion of Möbius invariant spaces of $\mathcal{H}$-harmonic functions and the Berezin transform on $\mathbb{B}$. We conclude the chapter with three theorems of Hardy and Littlewood for $\mathcal{H}$-harmonic functions, and the Littlewood–Paley inequalities for $\mathcal{H}$-subharmonic functions.

At the end of each chapter, I have included a set of exercises dealing with the topics discussed. Many of these problems involve routine computations and inequalities not included in the text. They also provide examples relevant to the topics of the chapter. Also included are problems whose solutions may be suitable for possible publication. The latter are marked with an asterisk.

Acknowledgments

The preliminary draft of these notes comprising most of Chapters 1–3 and parts of Chapters 5 and 6 was written while the author was on sabbatical leave at the CRM at the University of Montreal during fall 1999. Since 1999 there has been considerable research activity on hyperbolic function theory on $\mathbb{B}$. Included in this has been the work of P. Jaming [41], [42], [43], S. Grellier and P. Jaming [31], M. Jevtić [44], and the author [89], among others.

I would like to thank my good friend Paul Gauthier at the University of Montreal for providing the motivation for this project. His many questions about harmonic functions on real hyperbolic space, to which I did not know the answers, encouraged me to learn more about this interesting subject. I am also indebted to Paul for using the preliminary draft of these notes in one of his courses and for pointing out a number of errors. I would also like to express my appreciation to John Taylor at McGill for providing me with a copy of his notes on rank one symmetric spaces, and to thank the Director and Staff at the CRM for their hospitality during my visit. Finally, I would like to thank Wolfgang Woess Universität Graz, for bringing my original manuscript to the attention of Sam Harrison, Editor at Cambridge University Press, and the staff at Cambridge University Press and SPi Global for their assistance in the completion of the project.

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Contents

1

Möbius Transformations

In this chapter we provide a brief review of Möbius transformations on n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$). A good reference for these topics is the monograph by A. F. Beardon [11]. First, however, we begin with a review of notation that will be used throughout these notes.

1.1 Notation

For x, $y \in \mathbb{R}^n$ we let $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$ denote the usual inner product on $\mathbb{R}^n$ and $|x| = \sqrt{\langle x, x\rangle}$ the length of the vector x. For $a \in \mathbb{R}^n$ and $r > 0$, the ball $B(a, r)$ and sphere $S(a, r)$ are given respectively by

$$B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\},$$
$$S(a,r) = \{x \in \mathbb{R}^n : |x-a| = r\}.$$

The unit ball and unit sphere with center at the origin will simply be denoted by $\mathbb{B}$ and $\mathbb{S}$ respectively.[1] The **one-point compactification** of $\mathbb{R}^n$, denoted $\hat{\mathbb{R}}^n$, is obtained by appending the point ∞ to $\mathbb{R}^n$. A subset U of $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is open if it is an open subset of $\mathbb{R}^n$, or if U is the complement in $\hat{\mathbb{R}}^n$ of a compact subset C of $\mathbb{R}^n$. With this topology $\hat{\mathbb{R}}^n$ is compact.

For a subset D of $\mathbb{R}^n$, $\overline{D}$ denotes the closure of D, Int (D) the interior of D, ∂D the boundary of D, and $\widetilde{D}$ the complement of D in $\mathbb{R}^n$. Also if E and F are sets, $E \setminus F$ denotes the complement of F in E, that is, $E \setminus F = E \cap \widetilde{F}$.

The study of functions of n-variables is simplified with the use of multi-index notation. For an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_j is a non-negative integer, the following notational conventions will be used throughout:

[1] If we wish to emphasize the dimension n, we will use the notation $\mathbb{B}_n$ and $\mathbb{S}_n$ to denote the unit ball and sphere in $\mathbb{R}^n$.

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If Ω is an open subset of $\mathbb{R}^n$, we denote by $C^k(\Omega)$, $k = 0, 1, 2, \ldots$ the set of real-valued (or complex-valued) functions f on Ω for which $D^\alpha f$ exists and is continuous for all multi-indices α with $|\alpha| \leq k$. Thus $C^0(\Omega)$, or simply $C(\Omega)$, denotes the set of real-valued (or complex-valued) continuous functions on Ω, and $C^\infty(\Omega)$ the set of infinitely differentiable functions on Ω. Also, the set of functions $f \in C^k(\Omega)$ for which $D^\alpha f$, $|\alpha| \leq k$, has a continuous extension to $\overline{\Omega}$ will be denoted by $C^k(\overline{\Omega})$. If $f : \Omega \mapsto \mathbb{R}$, then the **support** of f, denoted $\operatorname{supp} f$, is defined as

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The set of continuous functions on Ω with compact support will be denoted by $C_c(\Omega)$. The notations $C_c^k(\Omega)$ and $C_c^\infty(\Omega)$ have the obvious meanings.

A linear transformation $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be **orthogonal** if $|Ax| = |x|$ for all $x \in \mathbb{R}^n$. The set of orthogonal transformations of $\mathbb{R}^n$ will be denoted by $O(n)$. If A is represented by the $n \times n$ matrix $(a_{i,j})$, then A is orthogonal if and only if

$$\sum_{k=1}^{n} a_{i,k} a_{j,k} = \delta_{i,j} = \begin{cases} 1 & i = j, \\ 0, & i \neq j. \end{cases}$$

If $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$ is a C^1 mapping of an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, then the derivative $\psi'(x)$ is the $n \times n$ matrix given by

$$\psi'(x) = \left(\frac{\partial \psi_i}{\partial x_j} \right)_{i,j=1}^{n},$$

and the **Jacobian** J_ψ of the transformation ψ is given by $J_\psi(x) = \det \psi'(x)$.

1.2 Inversion in Spheres and Planes

Definition 1.2.1 *The* ***inversion***[2] *(or reflection) in the sphere $S(a, r)$ is the function $\phi(x)$ defined by*

[2] Although we will mainly be interested in the case $n \geq 2$, the formulas for inversions in spheres and planes are still meaningful when $n = 1$.

$$\phi(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a). \qquad (1.2.1)$$

The inversion in the unit sphere $\mathbb{S}$ is the mapping $\phi(x) = x^*$ where

$$x^* = \begin{cases} \dfrac{x}{|x|^2} & x \neq 0, \infty, \\ 0 & x = \infty, \\ \infty & x = 0. \end{cases}$$

Thus (1.2.1) can now be rewritten as

$$\phi(x) = a + r^2(x-a)^*.$$

The reflection $\phi(x)$ is not defined at $x = a$. Since $|\phi(x)| \to \infty$ as $x \to a$ we set $\phi(a) = \infty$. Also, since $\lim_{|x|\to\infty} |\phi(x) - a| = 0$, we set $\phi(\infty) = a$. Thus ϕ is defined on all of $\hat{\mathbb{R}}^n$, and it is easily shown that ϕ is continuous in the topology of $\hat{\mathbb{R}}^n$. A straightforward computation also shows that $\phi(\phi(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Thus ϕ is a one-to-one continuous map of $\hat{\mathbb{R}}^n$ onto $\hat{\mathbb{R}}^n$ satisfying $\phi(x) = x$ if and only $x \in S(a, r)$.

In addition to reflection in a sphere we also have reflection in a plane. For $a \in \mathbb{R}^n$, $a \neq 0$, and $t \in \mathbb{R}$, the plane $P(a, t)$ is defined by

$$P(a, t) = \{x \in \mathbb{R}^n : \langle x, a\rangle = t\}.$$

By convention ∞ belongs to every plane $P(a, t)$.

Definition 1.2.2 *The* **inversion** *(or reflection) in the plane $P(a, t)$ is the function $\psi(x)$ defined by*

$$\psi(x) = x + \lambda a,$$

where $\lambda \in \mathbb{R}$ is chosen so that $\frac{1}{2}(x + \psi(x)) \in P(a, t)$.

Solving for λ gives

$$\psi(x) = x - 2[\langle x, a\rangle - t]a^*, \quad x \in \mathbb{R}^n. \qquad (1.2.2)$$

For the mapping ψ we have

$$|\psi(x)|^2 = |x|^2 + O(|x|),$$

and as a consequence $\lim_{|x|\to\infty} |\psi(x)| = \infty$. Thus as above we define $\psi(\infty) = \infty$. With this definition the mapping ψ again satisfies $\psi(\psi(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Thus ψ is a one-to-one continuous map of $\hat{\mathbb{R}}^n$ onto itself with $\psi(x) = x$ if and only if $x \in P(a, t)$. It is well known that each inversion (in a sphere or a plane) is orientation-reversing and conformal (see [11, Theorem 3.1.6]).