

# Probabilistic Methods in the Theory of Structures

**Strength of Materials, Random Vibrations,  
and Random Buckling**

Isaac Elishakoff



World Scientific

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Probabilistic Methods  
in the  
Theory of Structures  
**Strength of Materials, Random Vibrations,  
and Random Buckling**



To my father, Ben Zion Elishakoff, z"l  
To my mother, Leah (Margaret) Elishakoff, z"l



## Preface to the Third Edition

The first edition of this book appeared over three decades ago (Wiley-Interscience, 1983), whereas the second one saw light on the verge of new millennium (Dover, 1999). This is third, corrected and expanded edition that appears in conjunction with its companion volume *Probabilistic Methods in the Theory of Structures: Complete Worked-Through Solutions*.

Thus, the reader is able to both get acquainted with the theoretical material and be able to master some of the problems, following Chinese dictum: *What I hear, I forget; what I see, I may remember; what I experience, I know for life*.

The main idea of the book lies in the fact that three topics: probabilistic strength of materials, random vibrations, and probabilistic buckling are presented in a single package allowing one to see the forest in between the trees. Indeed, these three topics usually are presented in separate manners, in different specialized books. Here, the reader gets a feeling of true unity of the subject at large in order to appreciate that in the end what one wants is reliability of the structure, in conjunction with its operating conditions.

As I describe in the Preface to the second edition, this book was not conceived *ab initio*, as a book that author strived to compose. Rather, it was forced, as it were, upon me due to two reasons. One was rather a surprising but understandable requirement in the venerable Delft University of Technology, The Netherlands to prepare the lecture notes for students with the view of reducing skyrocketing costs of acquisition of textbooks by the students. The other one was an unusually warm acceptance of the notes that I prepared while at Delft University of Technology and later in Haifa, at the Technion-Israel Institute of Technology by the legendary engineering scientist Warner Tjardus Koiter (1914–1997). The energy necessary to prepare the second and third editions came from enthusiastic reviews that appeared in various sources (see the selection of these on the back cover).



Author embraced the simplicity of exposition as the main virtue following Isaac Newton's view that "*Truth is ever to be found in simplicity, and not in the multiplicity and confusion of things.*"

Isaac Elishakoff  
Boca Raton, May 2016

# Preface to the Second Edition

I am delighted that Dover Publications is presenting to readers *Probabilistic Theory of Structures*, the second, revised edition of my first book, *Probabilistic Methods in the Theory of Structures*. It is both an exciting and humbling event to have a book published in a series that also includes works by Lord Rayleigh and A. E. H. Love, amongst other classics, which have contributed to present-day theoretical and applied mechanics.

Perhaps this is the proper occasion for telling the story of how the first edition of this book came to be written, some twenty years ago. The idea for a book on probabilistic methods originated with the late Professor Alexander Kornecki, my then colleague at the Technion-Israel Institute of Technology. It was the fall of 1972, the year I emigrated from the former Soviet Union and was hired by the Technion as a Lecturer. Professor Kornecki, who was spending a sabbatical at Princeton University, wrote me expressing his opinion that existing books on probabilistic methods all seemed to start somewhere in the middle, reviewing the probability theory and random processes briefly and with such speed that no taste was left in the mouth of the hungry reader; or else they dispensed with introductions and started with the beef itself. My senior colleague urged me to write a book that could be not only read but also understood. I was greatly flattered by this suggestion, but regarded it as impracticable. As a brand-new hireling in a new-ancient country, I knew that one must first publish extensively (without perishing) and only then, given the desire and the energy, embark on such a monumental undertaking.

Six years later, having in the meantime mastered Hebrew, my new old language, and taught, along with numerous undergraduate courses, two specialized graduate courses—"Introduction to Probabilistic Methods" and "Random Vibrations"—I became entitled to a sabbatical leave. This

custom of taking an academic “time out” may seem a superfluous luxury at first glance, but, as my experience bears out, it provides a valuable opportunity for renewal and fruitful interaction with other environments and scientists. At any rate, following the Biblical concept of the sabbatical year, and availing myself of the generosity of the Technion, I started to look for an institution that would be interested in my research.

Fortunately, several universities in North America and Europe offered guest appointments for the 1979/80 academic year. We chose to go to Europe; I had always liked manageable distances, especially since my driving experience was limited to tiny Israel. Of the European institutions that were interested, I chose the Delft University of Technology, mainly because of my desire to conduct specific research on initial imperfection sensitivity, whose deterministic aspects had been so vividly uncovered at Delft some 35 years earlier by Professor Dr. Ir. Warner Tjardus Koiter in his famous Ph.D. dissertation, and then had been pursued analytically, numerically and experimentally—with great success—by Professor Dr. Johann Arbocz. I was dreaming of combining my recently developed probabilistic techniques with the numerical codes developed at Delft.

Upon my arrival, I found that Delft was to pose two unexpected challenges. First, Professor Arbocz himself told me that he did not believe in probability, though he said I was welcome to try to prove my own belief. Second, to my astonishment, Ir. Dijkshoorn appeared in my office during my first week there and said: “You will be returning to your home university in a year, but we would like you to leave some scientific trace here. Please prepare written lecture notes on probabilistic methods, so that future generations of students and researchers at our university can use them.” I was not enthusiastic about such a task. Not only had I been unaware of this traditional Delft requirement, but preparing the lecture notes would disrupt my plan to devote all of my time to research. However, thanks to several elements unique to Delft I was able to meet this challenge. These elements included the extremely friendly atmosphere created by Johann Arbocz, his wife Margot, and the entire “Vakgroep C” of the Aerospace Department. The warm hospitality of the Arbocz, Koiter, van der Neut, Schijve, van Geer and other families is unforgettable. The extremely fast and skillful typing of Marijke Schillemans and the beautiful artwork of draftsman Willem Spee were additional blessings. The exceptional situation these elements yielded would eventually produce active scientific cooperations between myself and J. Arbocz, A. Scheurkogel, J. Kalker, W. Verduyn, J. van Geer, T. van Baten, S. van Manen, P. Vermeulen, and in later years with W. T. Koiter.

Another beautiful Delft tradition I had not known about was the distribution of faculty members’ memoranda to faculty in other depart-

ments. This, too, was to take me in a direction I had not anticipated. In March 1980 a conference organized by W. T. Koiter took place at the Mechanical Engineering Department. Professor Koiter, as its chair, met every participant at the “Centraal Hotel.” He asked me if I would accompany him home, once all the participants had arrived. So we waited together until nearly midnight, and we had several glasses of *jonge jenever*. At that time, Professor Koiter told me that he had read my memorandum on probabilistic methods, which had been distributed to his department. He complimented me on its explanatory style and contents, and urged me to base a book on these materials. I was very excited. With this blessing from Professor Koiter (who was characterized by Bernard Budiansky—himself a champion of American mechanics—as “the sage from Delft”), my energies received a tremendous boost. I decided to devote the remaining months of my sabbatical to advancing this project as far as I could.

Within three years, the manuscript was completed at the Technion and had been accepted by Wiley. Upon the book’s initial publication I refrained from mentioning the fact that Professor Koiter’s enthusiastic encouragement was the main driving force behind the book. As I later explained to him, I had felt that I should not use his approval to promote my work, that the work must stand on its own feet. Fortunately, the book was well received by the reviewers as well as my colleagues. I especially cherish the reviews of Frank Kozin, Jim Yao, and John Robson, and the letters of Steve Crandall, Masanobu Shinozuka, Niels Lind, Günther Natke, and many others. The first edition was adopted as a textbook at, among many other places in the U. S. and elsewhere, Columbia University, where Professor Alfred Freudenthal had developed probabilistic approaches in mechanics, following his pioneering work at the Technion. Credit is due to this last institution which, with its unique system of teaching and research, of offering seminars in which almost every possible criticism is heard and the best challenges are posed to the presenter of new ideas, I consider to have been my true academic launcher. This environment helped me to search out the clearest presentation—one that, following the advice of the late Professor Kornecki, would enable both students and faculty to understand the material.

During the 1985/86 academic year, while serving as Frank Freimann Visiting Chair Professor at the University of Notre Dame, I found the time to prepare a detailed solutions manual for the text. (If you are an instructor who has adopted this text for your course and you would like to receive this manual, please contact me at the email address: [ielishak@me.fau.edu](mailto:ielishak@me.fau.edu).) Thanks are extended to Dr. Gabriel Cederbaum of the University of Beer-Sheva in the Negev who during this period provid-

ed a number of the solutions used in the manual, and painstakingly checked most of them; to Professor Michael Dimentberg who, while using the text at the Worchester Polytechnic, brought some misprints to my attention. I am also indebted to Professors Yehuda Stavsky and Menachem Baruch of the Technion-Israel Institute of Technology, for discussions on a number of probabilistic topics over the years.

I appreciate the cordial cooperation of John Grafton at Dover in the preparation of this second edition of the book. Heartfelt gratitude goes to Eliezer Goldberg, formerly of the Technion, for providing unmatched editorial assistance with 100% reliability. I am forever indebted to my great teacher Academician V. V. Bolotin of Moscow for his inspirations over the entire nine-year period that I spent at the Moscow Power Engineering Institute and State University, and for providing a superb model of the deepest forms of critical thinking. I also want to thankfully mention the exceptional educational system in Georgia, former Soviet Union, and especially my elementary school teacher Aneta Zekvava, who taught her students to love science and to be kind. I have been very fortunate in having wonderful family, teachers, and friends on many continents.

I recall our insightful secretary at the Technion, Mrs. Dvora Zirkin, asking me in the final stages of preparing the manuscript: "Why do you dedicate the book to your parents, rather than to your wife and children?" My response had been quick: "Because I met my parents earlier! I'll dedicate my second book to my wife and children." My parents taught me to admire appropriateness, humility and truth, while my wife and children provided love and tolerance without which no one could dream of completing a big project. Last but not least, thanks to my students at the Technion, at Delft University of Technology, at the University of Notre Dame, at the Naval Postgraduate School, and at Florida Atlantic University, who taught me how to teach them ever better.

Isaac Elishakoff  
Boca Raton, FL

September 1998

# Preface to the First Edition

This book is written both to serve as a first-course text on probabilistic methods in the theory of structures and to provide a more advanced treatment of random vibration and buckling. It is intended in particular for the student in aeronautical engineering, mechanical engineering, or theoretical and applied mechanics, and may also be used by practicing engineers and research workers as a reference. In fact, it combines the features of a textbook and a monograph.

Probability theory and random functions are playing an ever more prominent role in structural mechanics due to the growing realization that many mechanical phenomena can be satisfactorily described by probabilistic means only. In the last 25 years, much work has been done and many studies have been published on this subject. However, despite significant advances, the probabilistic approach to the theory of structures has not yet found its proper place in engineering education.

Chapter 1 introduces the role of probabilistic methods in the theory of structures. Chapters 2 through 4 deal exclusively with elements of the theory of probability for a single random variable. This apparent preoccupation with the single random variable stems from my own feeling that it would be unfair to offer the reader a mere taste of the theory of probability and then immediately confront him or her with a wide range of applications. Chapter 5 is devoted to the reliability of structures described by a single random variable. Chapter 6 discusses elements of the theory of probability of two or more random variables, while Chapter 7 examines the reliability of such multivariable structures. Chapter 8 introduces the theory of random functions. Chapter 9 deals with random vibration of single- and multidegree-of-freedom structures, and Chapter 10 with random vibration of continuous systems. These chapters concentrate on the role of modal cross correlations in random vibration analysis, usually overlooked in literature, as well as treat point-driven struc-

tures, and random vibration and flutter. These chapters constitute, among others, a prerequisite to study the fatigue life of structures—a topic which is outside the scope of this book. The reader interested in this subject is referred to other sources where it is adequately treated. Finally, Chapter 11 is devoted to the Monte Carlo method for treating problems incapable of exact solution. Special emphasis is placed on buckling of nonlinear structures where random imperfections may be responsible for drastic reduction of the buckling loads.

Ample examples are included in the book, because it is my experience that much of the material in question may be taught most effectively by this means. An additional purpose of the examples is to examine the validity of some widely accepted simplifying assumptions concerning the probabilistic nature of the output quantities and to observe the errors that these assumptions may cause. Numerous exercises are provided with each chapter, to deepen the reader's grasp of the subject and widen his or her perspectives.

The material in Chapters 1 to 5, together with Sections 11.1–11.3, are suitable for a one-semester, first-level course at the junior or senior level. Prerequisite courses for this part are calculus, differential equations, and mechanics of solids. For departments whose curriculum requires a course in the theory of probability, the first four chapters may be rapidly recapitulated, in which case the one-semester course may include also Chapters 6 and 7, as well as Sections 11.4 and 11.5. The material in Chapters 8 through 11 is open to both the analytically minded senior and the graduate student and may form an advanced course on random vibration and buckling. The additional prerequisite for this part is knowledge of matrix theory and the basics of vibration and buckling of structures, although the necessary material is reviewed at the beginning of each chapter.

It is my agreeable duty to thank the Department of Aerospace Engineering of Delft University of Technology for their invitation to present a series of lectures (from which this text grew) to their students and scientific staff during my sabbatical leave in the academic year 1979–1980—an experience of endless Dutch courtesy and good will. My sincere thanks are due to the Dean, Prof. Ir. Jaap A. van Ghesel Grothe, and to Professor of Aircraft Structures Dr. Johann Arbocz for their constant encouragement and help. Appreciation is expressed to the staff members and the students of Delft, and especially to Ir. Johannes van Geer, Ir. Willie Koppens, and Ir. Kees Venselaar for their able assistance in a number of calculations and constructive suggestions concerning the lecture notes. I acknowledge the help of Ir. J. K. Vrijling of Delft in writing Sec. 4.18. I also thank the Department of Aeronautical Engineering, Technion-Israel Institute of Technology, in whose encouraging atmosphere I was able to bring this work to completion. I am also most indebted to Eliezer Goldberg of Technion, for his kind help in editing the text, to Marijke Schillemans and Dvora Zirkin for typing much of the original manuscript, to Alice Aronson and Bernice Hirsch for typing Chapter 9, and to Willem Spee and Irith Nizan for preparing the drawings.

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