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Fixed Point Theory

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Editor's Preface

Approach your problems from the right end and begin with the answers. Then, one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. Van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G. K. Chesterton, *The Scandal of Father Brown* 'The Point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces.

This series of books, *Mathematics and Its Applications*, is devoted to such (new) interrelations as *exempla gratia*:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavor into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

With books on topics such as these, of moderate length and price, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

The present book furnishes good example of a central concept (technique) with multitudes of different uses. Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control, and functional analysis. Moreover, more or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician.

The unreasonable effectiveness of mathematics in science . . .

Eugene Wigner

Well, if you knows of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange

Krimpen a/d IJssel
March, 1979

Michiel Hazewinkel

Foreword

This book is intended as an introduction to fixed point theory and its applications. The topics treated range from fairly standard results (such as the Principle of Contraction Mapping, Brouwer's and Schauder's fixed point theorems) to the frontier of what is known, but we have not tried to achieve maximal generality in all possible directions. We hope that the references quoted may be useful for this purpose.

The point of view adopted in this book is that of functional analysis; for the readers more interested in the algebraic topological point of view we have added some references at the end of the book. A knowledge of functional analysis is not a prerequisite, although a knowledge of an introductory course in functional analysis would be profitable. However, the book contains two introductory chapters, one on general topology and another on Banach and Hilbert spaces. As a special feature of these chapters we note the study of measures of noncompactness; first in the case of metric spaces, and second in the case of Banach spaces.

Chapter 3 contains a detailed account of the Contraction Principle, perhaps the best known fixed point theorem. Many generalizations of the Contraction Principle are also included. We note here the connection between ideas from projective geometry and contractive mappings. After presenting some ways to compute the fixed points for contractive mappings, we discuss several applications in various areas.

Chapter 4 presents Brouwer's fixed point theorem, perhaps the most important fixed point theorem. After some historical notes concerning opinions about Brouwer's proof – which have been influential for the future of the fixed point theory (Alexander and Birkhoff and Kellogg) – we present many proofs of this theorem of Brouwer, of interest to different categories of readers. Thus we present an elementary one, which requires only elementary properties of polynomials and continuous functions; another uses differential forms; still another uses differential topology; and one relies on combinatorial topology. These different proofs may be used in different ways to compute the fixed points for mappings. In this connection, some algorithms for the computation of fixed points are given. The chapter ends with some applications, among which we mention here those concerning economic equilibrium prices.

Chapter 5 is a natural continuation of Chapter 4 and presents the generalizations of Brouwer's theorem obtained by Schauder as well as the new results connected with Schauder's generalization of Brouwer's theorem. Various important and interesting contributions due to Krasnoselskii, Rothe, Altman, Ky Fan, F. Browder and also results due to Darbo and Sadovskii are presented. The contributions of Darbo and Sadovskii concern a new direction of extending the classical fixed point theorems, namely, using the so-called measures of noncompactness. We also present in this chapter some applications, among them the proof by Lomonosov of the existence of nontrivial hyperinvariant subspaces for bounded linear operators on Banach spaces commuting with nonzero bounded linear compact operators.

In Chapter 6 we present some results about mappings which do not increase distance – the so called nonexpansive mappings. First we present some results on the extension of nonexpansive mappings, with a simple example to show that, without certain restrictions on the mappings or on the space, fixed points do not exist for nonexpansive mappings. Further we present some results about the existence of fixed points for nonexpansive mappings or related classes of mappings on certain classes of Banach spaces, as well as results about the convergence of the iterates. An example and a method for computing fixed points for such mappings close this chapter.

Chapter 7 discusses results about fixed points for sequences of mappings. First we give some results for the case of contraction mappings, and second, for condensing mappings.

In Chapter 8 we present the elements of a theory of duality mappings and their connections with monotonic and nonexpansive operators as well as some surjectivity theorems which have many and useful applications in the theory of partial differential equations.

Chapter 9 contains results centering around the Markov–Kakutani results, including a beautiful result of Ryll–Nardzewski, as well as some results about the connection between invariant means on semigroups and fixed point theorems.

As a natural extension of the theory of fixed points for single-valued mappings, in Chapter 10, the case of set-valued mappings is considered. First we give some results about the Pompeiu–Hausdorff metric and results about set-valued contraction mappings. Some results concerning the extension of Brouwer and Schauder as well as Tichonov results for set-valued mappings are included.

Probabilistic metric spaces were introduced by Karl Menger in 1942, and since then the interest about these spaces has been growing constantly. We present in Chapter 11 some results about fixed points for contractive mappings on probabilistic metric spaces (abbreviated as PM-spaces) as well as some results concerning measures of noncompactness on these spaces and fixed points for certain classes of mappings.

Finally Chapter 12 contains results concerning topological degree. First the case of finite dimensional spaces is treated, in which we have the so-called Brouwer's degree; next this degree is extended to certain perturbations of the identity operator on Banach spaces. We include also the famous example of Leray which shows that it is not possible to define the degree for arbitrary perturbations.

Next, we present briefly the results obtained by extending the degree concept to certain perturbation of the identity by k-set contraction mappings. Using some approximation theorems (which are also of independent interest) we prove uniqueness of the topological degree. Next we give some algorithms for the computation of Brouwer's topological degree based on Stenger's formula. Some applications of the degree are noted at the end of this chapter.

We have tried to indicate the original location of the various results we have learned, and the references (which contain in turn references to many earlier results) may be used to obtain further information; when a result is not ascribed to anyone we do not make any claim to originality.

We wish to acknowledge with thanks conversations and correspondence on Functional Analysis, Operator Theory and Fixed Point Theory from which we have benefitted. The author wishes to acknowledge especially his debt to Professor Michiel Hazewinkel for his interest in the work, as well as for the suggestions concerning the material contained in this book.

My appreciation goes also to the editors of the D. Reidel Publishing Company for their attention to this volume.

VASILE I. ISTRĂTESCU

Table of Contents

Chapter 1

Topological Spaces and Topological Linear Spaces

1.1. METRIC SPACES

The notion of convergence is of fundamental importance in analysis as well as in many chapters of mathematics. Thus we have the convergence of sequences of real numbers, the convergence of sequences of complex numbers, the convergence of sequences of functions, etc. It is worth remarking that in the case of functions we have many types of convergence, for example: pointwise convergence; uniform convergence; the convergence in measure, etc.

Also, when we define the convergence of a sequence of numbers or functions we use points which are 'near' to our points. This vague notion of 'nearness' can be made precise using some functions which are called generally 'metrics' or 'distances'.

DEFINITION 1.1.1. Let S be a nonempty set and $d: S \times S \to R$.

The function d is called a metric on S (or distance) iff the following properties hold:

1. $d(x, y) = 0$ iff $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in S$,
3. $d(x, z) \leqq d(x, y) + d(y, z)$ for all $x, y, z \in S$.

The number $d(x, y)$ is called the distance between x and y and the pair (X, d) is called a metric space. For simplicity we write X and we say that X is a metric space.

PROPOSITION 1.1.2. *For any $x, y \in S$, $d(x, y) \geqq 0$.*

Proof. Let x, y, z be arbitrary points in S. In this case we have

$$d(x, z) \leqq d(x, y) + d(y, z)$$

and thus for $x = z$ we obtain

$$0 \leqq d(x, y) + d(y, x) = 2\, d(x, y)$$

and the assertion follows.

If S_1 is a subset of a metric space S then we can define a metric on S_1, simply by the relation

$$d_1(x, y) = d(x, y)$$

and d_1 is called the induced metric on S_1.

We give now some examples of metric spaces.

Example 1.1.3. Let E_n(or R^n) $= \{x = (x_1, x_2, \ldots, x_n), x_i \in \mathbf{R}, \mathbf{R}$ the set of real numbers$\}$ and let d be defined as follows: if $y = (y_1, y_2, \ldots y_n)$ then

$$d(x, y) = \left(\sum_1^n |x_i - y_i|^p \right)^{1/p} = d_p(x, y).$$

where p is a fixed number in $[1, \infty)$. The fact that d is a metric follows from the well-known Minkowski inequality. Also another metric on S considered above can be defined as follows:

$$d(x, y) = \sup_i \{|x_i - y_i|\} = d_\infty(x, y).$$

Example 1.1.4. Let S be the set of all sequences of real numbers $x = (x_i)_1^\infty$ such that for some fixed $p \in [0, \infty)$

$$\sum_1^\infty |x_i|^p < \infty.$$

In this case, if $y = (y_i)$ is another point in S, we define

$$d(x, y) = \left(\sum |x_i - y_i|^p \right)^{1/p} = d_p(x, y).$$

and from Minkowski's inequality it follows that this is a metric on S.

Example 1.1.5. Let $S = L^2_{[0, 1]} = \{f, \int_0^1 |f|^2 \mathrm{d}t < \infty\}$ and for any two functions (classes of functions) f, g we define

$$d_2(f, g) = \left(\int_0^1 |f - g|^2 \, \mathrm{d}t \right)^{1/2}$$

and from Minkowski's inequality for integrals, it follows that this is a metric on S.

Example 1.1.6. Let $S = \mathbf{C}_{[0, 1]}$ be the set of all continuous complex-valued functions on $[0, 1]$. We define, for any f, g in S

$$d(f, g) = \sup \{|f(t) - g(t)| : t \in [0, 1]\}$$

and it is easy to see that this is a metric on S.

Example 1.1.7. For the set S of all real numbers, which in what follows will be denoted by **R**, we define

$$d(x, y) = |x - y|$$

for any two real numbers x, y. Then $(\mathbf{R}, d)$ is a metric space.

Example 1.1.8. Let S be the set **Q** of all rational numbers in **R** and the metric induced by d. Then $(\mathbf{Q}, d_1)$ is a metric space. The notion of convergence in metric spaces is defined as follows:

DEFINITION 1.1.9. A sequence $\{x_n\}$ in a metric space (X, d) converges to an element x of X if, for any $\varepsilon > 0$, there exists N_ε such that for all $n \geqq N_\varepsilon$,

$$d(x_n, x) \leqq \varepsilon.$$

An important class of sequences in metric spaces are the so-called Cauchy sequences. These sequences are defined as follows:

DEFINITION 1.1.10. A sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence if, for any $\varepsilon > 0$, there exists N_ε such that for all $n, m \geq N_\varepsilon$,

$$d(x_n, x_m) \leqq \varepsilon.$$

It is easy to see that any sequence which converges is a Cauchy sequence.

An important class of metric spaces in which the converse is also true is the class of so-called 'complete metric spaces' and formally this class is introduced in the following

DEFINITION 1.1.11. A metric space (X, d) in which any Cauchy sequence $\{x_n\}$ has the property that it converges to a point of X, is called complete.

Example 1.1.12. All the spaces in Examples 1.1.5–1.1.6 are complete metric spaces; the metric space in Example 1.1.7 is not complete.

For any $r > 0$ and $x \in X$, X a metric space, we define

1. $S_r(x) = \{y, d(x, y) \leqq r\}$ the disc with centre x and radius r,
2. $\mathring{S}_r(x) = \{y, d(x, y) < r\}$ the open disc with centre x and radius r,
3. $\partial S_r(x) = \{y, d(x, y) = r\}$ the boundary of the disc with centre x and radius r or the circumference of centre x and radius r.

DEFINITION 1.1.13. Let (X, d) be a metric space and G a subset of X. The point $x \in G$ is said to be interior to G if there exists an open disc $S_x(X) \subset G$. The set G is called open if all its points are interior points or is the empty set.

DEFINITION 1.1.14. A set F in a metric space is called closed if the set

$$C_F = \{x, x \in X, x \notin F\}$$

is an open set.

For any set in a metric space X, the diameter is the number

$$d(A) = \sup \{d(x, y), x, y \in A\}$$

and the distance from a point $x \in X$ to the set A is the number

$$d(x, A) = \inf \{d(x, y); y \in A\}.$$

We define now the fundamental notion of neighbourhood of a point in a metric space.

DEFINITION 1.1.15. If X is a metric space and $x \in X$ is an arbitrary point then a neighbourhood of x is any set which contains an open set containing x.

The important notion of continuity is introduced as follows:

DEFINITION 1.1.16. If X and Y are two metric spaces and $f: X \to Y$ is any function, then we say that f is continuous at $x \in X$ if, for any neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that for all $z \in U$, $f(z) \in V$. The function f is continuous on X if it is continuous at each point of X.

We have the following characterization of continuous functions at x:

THEOREM 1.1.17. *If X and Y are two metric spaces and $f: X \to Y$ is any function, then f is continuous at $x \in X$ if and only if (contracted to 'iff' following a suggestion of Halmos) for any sequence $(x_n) \subset X$ converging to x, the sequence $(f(x_n))$ converges to $f(x)$.*

Proof. Since the proof is the same as for functions defined on [0, 1] we omit it.

DEFINITION 1.1.18. If X is metric space and S is any set in S then the closure of S is the intersection of all closed sets containing S and is denoted by $\bar{S}$.

Example 1.1.19. There exist metric spaces for which

$$\bar{\mathring{S}}_r(x) \neq S_r(x).$$

For, if we take X as

$$X = \{x, 0 \leqq x \leqq 1\} \cup \{e^{i\theta}, 0 \leqq \theta \leqq \pi/2\},$$

then clearly $S_1(0) = X$ and

$$\mathring{S}_1(0) = \{x, 0 \leqq x < 1\}.$$

This gives that

$$\bar{\mathring{S}}_1(0) = \{x, 0 \leqq x \leqq 1\}.$$

For the connection between complete metric spaces and incomplete metric spaces (i.e. metric spaces which are not complete) we mention the following result, which can be proved exactly as for the case $X = \mathbf{Q}$ the set of all rational numbers:

First we give the following

DEFINITION 1.1.20. If (X, d) is a metric space then the subset X_1 is dense in X if for any x in X there exists a sequence (x_n), $x_n \in X_1$ such that

$$\lim d(x_n, x) = 0.$$

Then we have

THEOREM 1.1.21. *If (X, d) is an incomplete metric space then there exists a complete metric space $(X^{\sim}, d^{\sim})$ such that for some function f we have*

1. $f: X \to X^{\sim}$,
2. $d(x, y) = d^{\sim}(f(x), (f(y))$,
3. *$f(X)$ is dense in $X^{\sim}$.*

We recall that any function f with Property 2 is called an isometry.

1.2. COMPACTNESS IN METRIC SPACES. MEASURES OF NONCOMPACTNESS

As is well known, for any bounded set E of real numbers there exists, for any sequence $(x_n) \subset E$, a convergent subsequence (this is the so called Bolzano-Weierstrass theorem), and any closed and bounded set can be characterized as having the following equivalent properties:

1. E has the property that for every sequence $(x_n) \subset E$ there exists a convergent subsequence to an element of E,
2. For any family $(V_i)_{i \in I}$ of open sets such that $E \subset \bigcup_{i \in I} V_i$ there exist $i_1, \ldots, i_m$ such that $E \subset \bigcup_1^m V_{i_j}$ (this is the so called Lebesgue–Borel Lemma).