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Sergey D. Algazin, Igor A. Kijko

AEROELASTIC VIBRATIONS AND STABILITY OF PLATES AND SHELLS

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This book is dedicated to the memory of a prominent scientist in mechanics and our teacher, A. A. Ilyushin.

Preface

Vibrations of engineering structures, aircraft elements (wings, fins), and thin-walled structural elements occurring upon their interaction with gas flow (as a rule, air flow) are referred to as “flutter”. One has to distinguish three main types of such vibrations: the classical flutter, exemplified by vibrations of aircraft wings and fins; stall flutter, exemplified by vibrations of suspension bridges, tall stacks; and panel flutter, to which vibrations of thin-walled elements (plates, shallow shells) of aircraft or rockets at (for the most part) supersonic speeds belong.

The growth of scientific interest in these phenomena was especially pronounced in the 1930s because of developments in aviation. We quote Russian test pilot M. L. Gallay: “With new high-speed aircraft becoming available, a wave of mysterious and unexplained air accidents rolled over almost all the developed countries. Casual eyewitnesses who spotted these accidents from the ground in all the cases described nearly the same picture: the aircraft was flying absolutely normally with nothing alarming noticed, and then suddenly some unknown force, as if by explosion, destroyed the aircraft – and the next moment twisted debris, wings, fins, body, are falling to the ground ... All the eyewitnesses independently described what they saw as an explosion ... However, investigation did not confirm this version: no traces of explosives, soot, or any burnt material were found on the debris ... This new dangerous phenomenon was named “flutter”, but, if I remember correctly, it was Molière who said that a sick person does not get well sooner only because he knows what his illness is called in Latin”[1]. This is a description of classical flutter.

A dramatic example of stall flutter is the Tacoma Narrows Bridge catastrophe in the USA, in which a suspension bridge (span 854 m, width 11.9 m) collapsed in 1940; see description of this accident in the above-quoted book.

A classical example of panel flutter is plate vibration in supersonic gas flow. The solution of many particular problems of this class became possible after A. A. Ilyushin discovered in 1947 the law of plane sections in high supersonic speed aerodynamics, after which the problem of panel flutter for plates (and, later, shallow shells) was formulated in a closed (by that time) form, leading to the development of effective analytical research methods. This (and other) questions are discussed in this book.

When writing this book, we did not aim to encompass or generalize the extensive bibliography on panel flutter available today (more than 700 works have been published on the subject since the 1930s). The main purpose was different: within the framework of mathematical models of the phenomenon which have been developed up till now, to present analytical and efficient numerical methods by which dif-

1 Quoted from: Ya. G. Panovko and I. I. Gubanova. Stability and Vibrations of Elastic Systems. Moscow, Fizmatlit, 1964, pp. 251–252.

ferent classes of panel flutter problems can be solved for plates and shallow shells. For this reason, only a few particular examples are considered in the book; we give preference to new problem formulations, mathematical substantiation of the research methods developed, and clarification of new mechanical effects. Some aspects of the approach, especially mathematical, have not yet been well-developed; we have noted some such aspects within the text, while others can be noticed by the thoughtful reader. We would greatly appreciate any comments with respect either to its content, or to possible further developments.

We hope that this book will be of interest to everyone involved in the analysis of dynamic stability of thin-walled structures.

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Introduction

In this book, vibrations of plates and shallow shells interacting with air flow are considered. As a rule, the main problem to be solved is to find out the parameter domain in which the vibrations are stable. The geometry and mechanical properties of the vibrating structural element are usually known; therefore, the question is the determination of flow velocity beyond which vibrations become unstable. The phenomenon of possible vibration instability is referred to as panel flutter, and the corresponding velocity is known as the critical flutter velocity.

The panel flutter problem became of particular interest during the post-war years of the 20th century, due to rapid development of aerospace engineering. Theoretical progress was promoted by the discovery of the law of plane sections in the aerodynamics of high supersonic velocities, which enabled, generally speaking, the coupled aeroelasticity problem to be resolved by a simple formula of the "piston theory".

The first studies relying on the piston theory were performed in the 1950s by A. A. Movchan et al. They considered the flutter problem for a rectangular plate in the simplest case where the flow velocity vector lies in the plane of the plate and is parallel to one of its edges. If asymptotic stability is of interest (which has been the case in the majority of past and current flutter studies), the problem is reduced to the analysis of the dependence of spectrum of a non-self-adjoint operator of the fourth order (its main part is the biharmonic operator) on the flow velocity. Evidently, even in this simplest formulation the flutter problem is far from trivial. Nevertheless, A. A. Movchan et al. obtained results which highlighted the essential points of the problem and, therefore, for a long time were considered as reference solutions.

Further development of approaches to flutter problem which followed these fundamental results did not touch upon the basic points of the theory: the forces of aerodynamic interaction of flow and vibrating element were described by the piston theory formula even in the cases where its applicability is questionable (a dramatic example is the flutter of a conical shell subjected to internal gas flow at high supersonic speed). At the same time, no attempts were made to formulate the flutter problem for a plate or shallow shell of arbitrary plan view shape; and such mathematical aspects as existence of the solution, general properties, structure of the spectrum, etc., were not touched upon. The large number of papers published in that period is attributed to the consideration of a variety of boundary condition combinations, physical effects of different nature (temperature, electromagnetic field), mechanical properties (viscoelastic, multilayered, anisotropic plates and shells), etc. The situation changed in the mid-1990s. On the one hand, new statements of flutter problems for plates and shells as parts of the aircraft cover at high supersonic velocities were formulated. On the other hand, a numerical-analytical nonsaturating method was developed which enabled an efficient solution of eigenvalue problems for non-self-adjoint operators (or systems of such operators) arising in the flutter studies. Taken together, these achievements allowed the class of flutter problems to be significantly extended and new mechanical effects to be obtained. These results, belonging, for the most part, to the authors and their colleagues, comprise the content of this book.

When presenting the material, as a rule we do not make reference to the original works and their authors. However, each part of the book begins with a brief introduction which indicates which works form the basis of the respective material.

Part I: **Flutter of plates**

The flutter problem for a rectangular plate was first formulated and studied to large extent by A. A. Movchan in the 1950s [495–498]. These results, today regarded as classical, became possible after A. A. Ilyushin discovered in 1947 the law of plane sections in high-speed aerodynamics [310]. One of its consequences is the local formula of the piston theory for additional pressure exerted by gas onto the vibrating plate, by which the flutter problem is reduced to an eigenvalue problem for a non-self-adjoint operator. A. A. Movchan et al. considered the problem in a rather limited particular case, where the flow velocity vector is parallel to one of the plate sides; numerous further works by other authors were devoted to a rather straightforward extension of the theory onto multilayered plates, account for the effects of different physical fields, etc.

The situation changed in mid-1990s when new formulations of panel flutter problems were presented by A. A. Ilyushin and I. A. Kijko [311]. Based on these formulations, the general spectral properties of the flutter operator were obtained, a numerical-analytical method was developed which enabled the spectrum to be studied, new classes of problems were solved, and new mechanical effects were discovered (I. A. Kijko and S. A. Algazin) [33–39, 41]. These results form the basis of the material presented in this section.

Also presented are some particular results on the flutter of plates of variable thickness or stiffness, as well as one special case of the optimization problem (V. I. Isaev and A. K. Kadyrov) [316, 330]. A new solution for the flutter of viscoelastic plate is given which resolves the long-existing critical flutter velocity paradox [350, 356].

1 Statement of the problem

Consider a plate occupying a domain S on the x, y plane, bounded by the contour Γ (hereafter, Γ is supposed to be piecewise-smooth). One side of the plate is subjected to gas flow with the velocity vector $\mathbf{v} = \{v_x, v_y\} = \{v\cos\theta, v\sin\theta\}$, $v = |\mathbf{v}|$. If, in addition to the unperturbed state $w_0 \equiv 0$, we consider a perturbation $w = w(x, y, t)$, then the aerodynamic pressure Δp caused by interaction with the perturbed flow appears. It will be shown in the following analysis that Δp is a linear operator of w, which will allow us to present the solution in the form $w = \varphi(x, y)\exp(\omega t)$, $\Delta p = \Delta p_0(x, y)\exp(\omega t)$ in all cases except the flutter problem for a viscoelastic plate.

The equation for vibrations of a constant-thickness plate takes the form

$$D\Delta^2 w + \rho h \frac{\partial^2 w}{\partial t^2} = \Delta p, \tag{1.1}$$

where $D = Eh^3/(12(1-\nu^2))$ is the cylindrical rigidity, E, ν, and ρ are Young's modulus, Poisson's ratio, and density of plate material, and h is its thickness. From the above considerations, we have $\Delta p_0 = L_1(\varphi) + L_2(\varphi, \omega)$, and therefore, (1.1) can be rewritten in the form

$$D\Delta^2\varphi + L_1(\varphi) + \rho h w^2\varphi + L_2(\varphi; \omega) = 0. \tag{1.2}$$

On the contour Γ, the deflection amplitude $\varphi(x, y)$ satisfies the boundary conditions

$$x, y \in \Gamma, \quad M_1(\varphi) = 0, \quad M_2(\varphi) = 0, \tag{1.3}$$

where the boundary operators M_1 and M_2 are problem-specific and will be given in each particular case. We assume hereafter that the plate is not subjected to any loads in its median plane.

The system of equations (1.2) and (1.3) represents a complicated eigenvalue problem with a non-self-adjoint operator; its eigenvalues are denoted by ω. By definition, we take that the perturbed motion of the plate is stable if $\operatorname{Re}\omega < 0$, and unstable if $\operatorname{Re}\omega > 0$; the critical parameters of the system (plate, flow) are determined by the condition $\operatorname{Re}\omega = 0$. In the further analysis, we consider the following main questions: determination of Δp, formulation of new problems; development of an efficient analytical approach, and identification of new mechanical effects.

2 Determination of aerodynamic pressure

Numerous studies on the vibrations and stability of plates in supersonic high-speed flows are carried out on the basis of the piston theory for the aerodynamic pressure Δp caused by the interaction of the flow and vibrating plate. This formulation has become so common that it was applied even in the cases where its validity is questionable. Here, we derive Δp in the cases of "moderate" supersonic ($M \sim 1.5$–2) and low subsonic velocities.

Consider an elastic strip occupying domain $S : \{0 \le x \le l, y = 0, |z| < \infty\}$. On the side $y \ge 0$, the strip is placed in a gas flow with unperturbed parameters (planar problem) $\mathbf{v} = \{u_0, 0\}, p_0, \rho_0, a_0 = (\gamma p_0/\rho_0)^{1/2}$, so that the unperturbed flow potential is $\varphi_0 = u_0 x$. Small vibrations of the strip $w(x,t)$ (with $w/\ell \ll 1$) cause flow perturbation; the perturbed flow potential is denoted by $\varphi_1 = \varphi_0 + \varphi$. Then we proceed in the usual way: from the Cauchy–Lagrange integral, equations of motion, mass conservation, and equation of state we obtain an equation for φ_1 and linearize it with respect to the perturbation φ to obtain

$$\frac{1}{a_0^2}\frac{\partial^2\varphi}{\partial t^2} + \left(M^2 - 1\right)\frac{\partial^2\varphi}{\partial x^2} + 2\frac{M}{a_0}\frac{\partial^2\varphi}{\partial x\partial t} - \frac{\partial^2\varphi}{\partial y^2} = 0, \tag{2.1}$$

where $M = u_0/a_0$. The potential φ must vanish at infinity and satisfy the impermeability condition on the line $y = 0$:

$$y = 0, \quad 0 \le x \le l, \quad \frac{\partial\varphi}{\partial y} = \frac{\partial w}{\partial t} + u_0\frac{\partial w}{\partial x} \tag{2.2}$$

$$y = 0, \quad x \le 0, \quad x \ge l, \quad \frac{\partial\varphi}{\partial y} = 0. \tag{2.3}$$

The overpressure in the flow is obtained from

$$p = -\rho_0\left(\frac{\partial\varphi}{\partial t} + u_0\frac{\partial\varphi}{\partial x}\right). \tag{2.4}$$

We search for the solution in the class of functions $\varphi(x,y,t) = f(x,y)\exp(\omega t)$, $w(x,t) = W(x)\exp(\omega t)$, and $p(x,y,t) = q(x,y)\exp(\omega t)$. Introduce now the nondimensional coordinates x/l and y/l, retaining hereafter the previous notation. Also, introduce the nondimensional frequency $l\omega/a_0 = \Omega$. The system of equations (2.1)–(2.4) is transformed to

$$(M^2 - 1)\frac{\partial^2 f}{\partial x^2} + 2M\Omega\frac{\partial\varphi}{\partial x} + \Omega^2 f - \frac{\partial^2 f}{\partial y^2} = 0 \tag{2.5}$$

$$y = 0, \quad 0 \le x \le 1, \quad \frac{\partial f}{\partial y} = a_0\left(\Omega W + M\frac{\partial W}{\partial x}\right) \tag{2.6}$$

$$y = 0, \quad x \le 0, \quad x \ge 1, \quad \frac{\partial\varphi}{\partial y} = 0 \tag{2.7}$$

$$q = -\frac{\rho_0 a_0}{l}\left(\Omega f + M\frac{\partial f}{\partial x}\right). \tag{2.8}$$

In what follows, it is necessary to distinguish the cases of $M < 1$ and $M > 1$; we consider them one by one.

For $M > 1$, perturbations are absent to the left of the point $x = 0$; therefore, it is possible to apply the Laplace transform along the x coordinate; condition (2.7) is not relevant, and the function $W(x)$ can be prolonged to $x \geq 1$ arbitrarily (as long as applicability conditions for the Laplace transform are satisfied), and this will not affect the overpressure $q(x, 0)$ acting on the strip. From (2.5) we obtain for the Laplace transform $\tilde{f}(s, y)$

$$\beta^2 \tilde{f} - \frac{\partial^2 \tilde{f}}{\partial y^2} = 0, \quad \beta^2 = \left(M^2 - 1\right) s^2 + 2M\Omega s + \Omega^2.$$

A solution bounded at infinity is

$$\tilde{f} = c_1 e^{-\beta y}. \tag{2.9}$$

From the boundary condition (2.8) for Laplace transform

$$\left.\frac{\partial \tilde{f}}{\partial y}\right|_{y=0} = -\beta c_1 = a_0(\Omega + Ms)\tilde{W}$$

it is possible to determine the parameter c_1, and therefore it follows from (2.9) that

$$\tilde{f} = -a_0 \frac{\Omega + Ms}{\beta} \tilde{W} e^{-\beta y}. \tag{2.10}$$

The overpressure (in terms of Laplace transforms) is now obtained from equation (2.8):

$$\tilde{q}(s, 0) = \Delta\tilde{p}(s) = \frac{\rho_0 a_0^2}{l} \frac{(\Omega + Ms)^2}{\beta} \tilde{W}(s). \tag{2.11}$$

The inverse Laplace transform is found from tables and convolution theorem. We first write $\beta = \sqrt{M^2 - 1}\sqrt{(s + s_1)(s + s_2)} \equiv \sqrt{M^2 - 1}\beta_0$; $s_1 = \Omega/(M-1)$, $s_2 = \Omega/(M + 1)$; $(s_1 + s_2)/2 = M\Omega/\left(M^2 - 1\right) \equiv \alpha_1$; $(s_1 - s_2)/2 = \Omega/\left(M^2 - 1\right) \equiv \alpha_2$.

We now have

$$L^{(-1)}\left(\frac{1}{\beta_0}\right) = I_0(\alpha_2 x) e^{-\alpha_1 x} \equiv H(x),$$

where $I_0(z)$ is the modified Bessel function; therefore

$$L^{(-1)}\left(\frac{\tilde{W}}{\beta_0}\right) = \int_0^x H(x - \tau) W(\tau)\, d\tau$$

$$L^{(-1)}\left(\frac{s\tilde{W}}{\beta_0}\right) = \int_0^x H(x - \tau) \frac{\partial W}{\partial \tau}\, d\tau$$

$$L^{(-1)}\left(\frac{s^2\tilde{W}}{\beta_0}\right) = \frac{\partial}{\partial x} \int_0^x H(x - \tau) \frac{\partial W}{\partial \tau}\, d\tau.$$

We perform the necessary calculations and substitute the results in equation (2.11) to obtain finally

$$
\begin{aligned}
\Delta p(x) = \frac{\rho_0 a_0^2 M}{l\,(M^2-1)^{1/2}} \Bigg[& \frac{M^2-2}{M^2-1}\Omega W + M\frac{\partial W}{\partial x} \\
& + \frac{\left(M^2+2\right)\Omega^2}{2M\left(M^2-1\right)^2}\int_0^x e^{-\alpha_1(x-\tau)} I_0\left(\alpha_2(x-\tau)\right) W(\tau)\,d\tau \\
& - \frac{2\Omega^2}{\left(M^2-1\right)^2}\int_0^x e^{-\alpha_1(x-\tau)} I_1\left(\alpha_2(x-\tau)\right) W(\tau)\,d\tau \\
& + \frac{M\Omega^2}{2\left(M^2-1\right)^2}\int_0^x e^{-\alpha_1(x-\tau)} I_2\left(\alpha_2(x-\tau)\right) W(\tau)\,d\tau \Bigg],
\end{aligned}
\tag{2.12}
$$

where $I_\nu(z)$, $\nu = 1, 2$, are the modified Bessel functions.

There are important implications of equation (2.12):

1. The formula of the piston theory is obtained in the limit $M \gg 1$, and it is valid only for the calculation of a few first eigenvalues Ω_n such that $|\Omega_n|/M^2 \sim 1$ (or $|\alpha_2| \sim 1$), because $I_\nu(z)$ grow exponentially with increasing argument. This important point has not been taken into account so far.
2. If $|z| < 1$, then $I_\nu(z) \sim (z/2)^\nu$, therefore for "moderately" supersonic velocities $M^2 > 2$ the first few eigenvalues Ω_n can be calculated with the last two integral terms in equation (2.12) omitted and, also, with $\Delta p(x)$ taken in the form

$$
\begin{aligned}
\Delta p(x) \cong \frac{\rho_0 a_0^2 M}{\ell\sqrt{M^2-1}} \Bigg[& \frac{M^2-2}{M^2-1}\Omega W + M\frac{\partial W}{\partial x} \\
& + \frac{\left(M^2+2\right)\Omega^2}{2M(M^2-1)^2}\int_0^x e^{-\alpha_1(x-\tau)} W(\tau)\,d\tau \Bigg]
\end{aligned}
\tag{2.13}
$$

Assume now that the plate occupies a domain S on plane x, y with a boundary Γ, and it is subjected to a gas flow with velocity $\mathbf{v} = v_0\mathbf{n}_0 = \{v_0\cos\theta, v_0\sin\theta\}$. We assume that the overpressure $\Delta p(x, y)$ can be expressed by a formula which generalizes equation (2.13) (and, accordingly, (2.12)):

$$
\begin{aligned}
\Delta p(x,y) = \frac{\rho_0 a_0^2 M}{l\sqrt{M^2-1}} \Bigg[& \frac{M^2-2}{M^2-1}\Omega W + M\mathbf{n}^0 \operatorname{grad} W \\
& + \frac{\left(M^2+2\right)\Omega^2}{2M\left(M^2-1\right)^2}\int_{AB} e^{-\alpha_1(s-\tau)} W\left(x(\tau), y(\tau)\right)\,d\tau \Bigg],
\end{aligned}
\tag{2.14}
$$