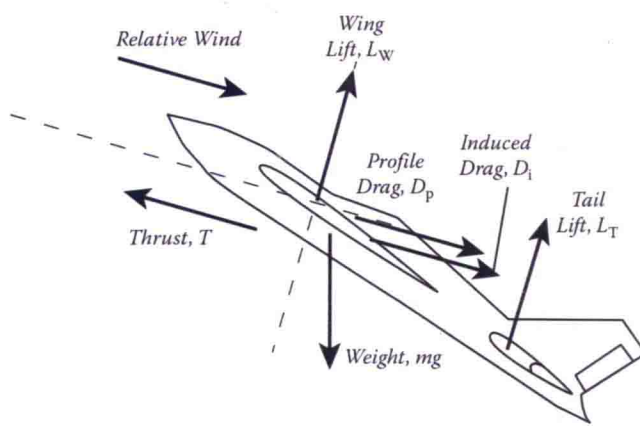


# Nonlinear Control of Robots and Unmanned Aerial Vehicles

## An Integrated Approach

Ranjan Vepa

 CRC Press  
Taylor & Francis Group



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# Nonlinear Control of **Robots** and **Unmanned Aerial Vehicles**

*To my parents, Narasimha Row and Annapurna*

# Preface

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During the last two decades, much progress has been made in the application of nonlinear differential geometric control theory, first to robotic manipulators and then to autonomous vehicles. In fact, robot control is simply a metaphor for nonlinear control. The ability to transform complex nonlinear systems sequentially to simpler prototypes, which can then be controlled by the application of Lyapunov's second method, has led to the development of some novel techniques for controlling both robot manipulators and autonomous vehicles without the need for approximations. More recently, a synergy of the technique of feedback linearization with classical Lyapunov stability theory has led to the development of a systematic adaptive backstepping design of nonlinear control laws for systems with unknown constant parameters. Another offspring of the Lyapunov-based controllers is a family of controllers popularly known as sliding mode controls. Currently, sliding mode controls have evolved into second- and higher-order implementations, which are being applied extensively to robotic systems.

Some years ago, the author embarked on a comprehensive programme of research to bring together a number of techniques in an attempt to formulate the dynamics and solve the control problems associated with both robot manipulators and autonomous vehicles, such as unmanned aerial vehicles (UAVs), without making any approximations of the essentially nonlinear dynamics. A holistic approach to the two fields have resulted in new application ideas such as the morphing control of aerofoil sections and the decoupling of force (or flow) and displacement control loops in such applications. A number of results of several of these studies were also purely pedagogical in nature. Pedagogical results are best reported in the form of new learning resources, and for this reason, the author felt that the educational outcomes could be best communicated in a new book. In this book, the author focuses on control and regulation methods that rely on the techniques related to the methods of feedback linearization rather than the more commonly known methods that rely on Jacobian linearization. The simplest way to stabilize the zero dynamics of a nonlinearly controlled system is to use, when feasible, input-output feedback linearization. The need for such a book arose due to the increasing appearance of both robot manipulators and UAVs with operating regimes involving large magnitudes of state and control variables in environments that are not generally very noisy. The underpinning themes which serve as a foundation for both robot dynamics and UAVs include Lagrangian dynamics, feedback linearization and Lyapunov-based methods of both stabilization and control. In most applications, a combination of these fundamental techniques provides a powerful tool for designing controllers for a range of application tasks involving tracking, coordination and motion control. Clearly, the focus of these applications is primarily on the ability to handle the nonlinearities rather than dealing with the environmental disturbances and noise which are of secondary importance. This book is of an applied nature and is about *doing* and *designing* control laws. A number of application examples are included to facilitate the reader's learning of the art of nonlinear control system design.

The book is not meant to supplant the many excellent books on nonlinear and adaptive control but is designed to be a complementary resource. It seeks to present the methods of nonlinear controller synthesis for both robots and UAVs in a single, unified framework.

The book is organized as follows: Chapter 1 deals with the application of the Euler–Lagrange method to robot manipulators. Special consideration is given to rapidly determining the equations of motion of various classes of manipulators. Thus, the manipulators are classified as parallel and serial, as Cartesian and spherical and as planar, rotating planar and spatial, and the methods of determining the equations of motion are discussed under these categories. The definition of planar manipulators is generalized so that a wider class of manipulators can be included in this category. The methods of deriving the dynamics of the manipulators can be used as templates to derive the dynamics of any manipulator. This approach is unique to this book. Chapter 2 focuses on the application of the Lagrangian method to UAVs via the method of quasi-coordinates. It is worth remembering that the use of the Lagrangian method for deriving the equations of motion of a UAV is not the norm amongst flight dynamicists. Moreover, the chapter introduces the velocity axes, as the synthesis of the flight controller in these axes is a relatively easy task. The concept of feedback linearization is introduced in Chapter 3, while the classical methods of phase plane analysis of the stability of nonlinear systems and their features are discussed in Chapter 4 in the context of Lyapunov’s first method. Chapter 5 presents an overview of the methods of robot and UAV control. Chapter 6 is dedicated to introducing the concepts of stability, and Chapter 7 is exclusively about Lyapunov stability with an enunciation of Lyapunov’s second method. The methodology of computed torque control is the subject of Chapter 8, and sliding mode controls are introduced in Chapter 9. Chapter 10 discusses parameter identification, including recursive regression, while adaptive and model predictive controller designs are introduced in Chapter 11. In a sense, linear optimal control, a particular instance of the Lyapunov design of controllers, is also covered in the section on model predictive control, albeit briefly. Chapter 12 is exclusively devoted to the Lyapunov design of controllers by backstepping. Chapter 13 covers the application of feedback linearization in the task space to achieve decoupling of the position and force control loops, and Chapter 14 is devoted to the applications of nonlinear systems theory to the synthesis of flight controllers for UAVs.

It is the author’s belief that the book will not be just another text on nonlinear control but serve as a unique resource to both the robotics and UAV research communities in the years to come and as a springboard for new and advanced projects across the globe.

First and foremost, I thank Jonathan Plant, without his active support, this project would not have been successful. I also thank my colleagues and present and former students at the School of Engineering and Material Science at Queen Mary University of London for their assistance in this endeavour. In particular, I thank Professor Vassili Toropov for his support and encouragement.

I thank my wife Sudha for her love, understanding and patience. Her encouragement and support provided the motivation to complete the project. I also thank our children Lullu, Satvi and Abhinav for their understanding during the course of this project.

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# Author

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**Ranjan Vepa, PhD**, earned a PhD (applied mechanics) from Stanford University (California), specializing in the area of aeroelasticity under the guidance of the late Professor Holt Ashley. He is currently a lecturer in the School of Engineering and Material Science, Queen Mary University of London, where since 2001, he has also been the director of the Avionics Programme. Prior to joining Queen Mary, Dr. Vepa was with the NASA Langley Research Center, where he was awarded a National Research Council Fellowship and conducted research in the area of unsteady aerodynamic modeling for active control applications. Subsequently, he was with the Structures Division of the National Aeronautical Laboratory, Bangalore, India, and the Indian Institute of Technology, Chennai, India.

Dr. Vepa is the author of five books: *Biomimetic Robotics* (Cambridge University Press, 2009); *Dynamics of Smart Structures* (Wiley, 2010); *Dynamic Modeling, Simulation and Control of Energy Generation* (Springer, 2013) and *Flight Dynamics, Simulation, and Control: For Rigid and Flexible Aircraft* (CRC Press, 2014). Dr. Vepa is a member of the Royal Aeronautical Society, London; the Institute of Electrical and Electronic Engineers, New York and the Royal Institute of Navigation, London. He is also a Fellow of the Higher Education Academy and a chartered engineer.

In addition, Dr. Vepa is studying techniques for automatic implementation of structural health monitoring based on observer and Kalman filters. He is involved in the design of crack detection filters applied to crack detection and isolation in aeroelastic aircraft structures such as nacelles, casings, turbine rotors and rotor blades for health monitoring and control. Elastic wave propagation in cracked structures is being used to develop distributed filters for structural health monitoring. Feedback control of crack propagation and compliance compensation in cracked vibrating structures are also being investigated. Another issue is the modeling of damage in laminated composite plates, nonlinear flutter analysis and the interaction with unsteady aerodynamics. These research studies contribute to the holistic design of vision-guided autonomous UAVs, which are expected to be used extensively in the future.

Dr. Vepa's research interests also include the design of flight control systems, aerodynamics of morphing wings and bodies with applications in smart structures, robotics, biomedical engineering and energy systems, including wind turbines. In particular, his focus is on the dynamics and robust adaptive estimation and control of linear and nonlinear aerospace, energy and biological systems with parametric and dynamic uncertainties. The research in the area of the aerodynamics of morphing wings and bodies is dedicated to the study of aerodynamics and its control, including the use of smart structures and their applications in the flight control of air vehicles, jet engines, robotics and biomedical systems. Other applications are in wind turbine and compressor control, maximum power point tracking, flow control over smart flaps and the control of biodynamic systems.



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# Lagrangian methods and robot dynamics

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## Introduction

The basis of the Newtonian approach to dynamics is the Newtonian viewpoint, that motion is induced by the action of forces acting on particles. This viewpoint led Sir Isaac Newton to formulate his celebrated laws of motion. In the late 1700s and early 1800s, a different view of dynamic motion began to emerge. According to this view, particles do not follow trajectories because they are acted upon by external forces, as Newton proposed. Instead, amongst all possible trajectories between two points, they choose the one which minimizes a specific time integral of the difference between the kinetic and the potential energies called the action. Newton's laws are then obtained as a consequence of this principle, by the application of variational principles in minimizing the action integral. Also, as a consequence of the minimization of the action integral, the total potential and kinetic energies of systems are conserved in the absence of any dissipative forces or forces that cannot be derived from a potential function. The alternate view of particle motion then led to a newer approach to the formulation and analysis of the dynamics of motion. It was no longer required to isolate each and every particle or body and forces acting on them, within a system of particles or bodies. The system of particles could be treated in a holistic manner without having to identify the forces of interaction between the particles or bodies.

The variational approach seeks to derive the equations of motion for a system of particles in the presence of a potential force field as a solution to a minimization problem. The independent variable in the problem will clearly be time, and the dependent functions will be the three-dimensional (3D) positions of each particle. The aim is to find a function  $L$  such that the paths of the particles between times  $t_1$  and  $t_2$  extremize the integral:

$$I = \int_{t_1}^{t_2} L(x, y, z, \dot{x}, \dot{y}, \dot{z}; t) dt. \quad (1.1)$$

The integral  $I$  will be referred to as the action of the system and the function  $L$  as the Lagrangian. In fact, we can show that when the Lagrangian  $L$  is defined as

$$L = T - V = \frac{1}{2}mv^2 - V(x, y, z; t), \quad (1.2)$$

the equations of motion are given by the Euler–Lagrange equations which are obtained by setting the variation of the Lagrangian  $\delta L$  to zero. Thus, we set

$$\delta L = \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial z} \delta z + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial \dot{z}} \delta \dot{z} = 0. \quad (1.3)$$

However, by expressing  $\delta L$  as

$$\delta L = \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) \right) \delta x + \left( \frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) \right) \delta y + \left( \frac{\partial L}{\partial z} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) \right) \delta z = 0, \quad (1.4)$$

and assuming that the variations  $\delta x$ ,  $\delta y$  and  $\delta z$  can be varied  $t$  without placing any constraints on them, it follows that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad (1.5)$$

with  $q_1 = x$ ,  $q_2 = y$  and  $q_3 = z$ . These are the celebrated Euler–Lagrange equations which result in Newton's second laws of motion when  $L = T - V$ .

Our focus in this chapter is the application of Lagrangian dynamics, not to particles in motion but to kinematic mechanisms in general, and robot manipulators in particular. To this end, a brief review of the kinematics of robot manipulators is essential.

## 1.1 Constraining kinematic chains: Manipulators

The primary element of a mechanical system is a link. A link is a rigid body that possesses at least two nodes that are points for attachment to other links. A joint is a connection between two or more links at specific locations known as their nodes, which allows some motion, or potential motion, between the connected links. A kinematic chain is defined as an assemblage of links and joints, interconnected in a way to provide a controlled output motion in response to a specified input motion. A mechanism is defined as a kinematic chain in which at least one link has been 'grounded', or attached, to a frame of reference which itself may be stationary or in motion. A robot manipulator is a controlled mechanism, consisting of multiple segments of kinematic chains, that performs tasks by interacting with its environment. Joints are also known as kinematic pairs and can be classified as a lower pair to describe joints with surface contact while the term *higher pair* is used to describe joints with a point or line contact. Of the six possible lower pairs, the revolute and the prismatic pairs are the only lower pairs usable in a planar mechanism. The screw, cylindrical, spherical and flat lower pairs are all combinations of the revolute and/or prismatic pairs and are used in spatial (three-dimensional) mechanisms.

### Manipulator kinematics: The Denavit and Hartenberg (DH) parameters

A primary problem related to the kinematics of manipulators is the forward kinematics problem, which refers to the determination of the position and orientation of the *end effector*, given the values for the joint variables of the robot. In the robotics community, a systematic procedure for achieving this in terms of four standardized parameters of a link, namely the joint angle, the link length, the link offset and the link twist, is adopted. This convention is known as the Denavit and Hartenberg convention, and the parameters are known as the Denavit and Hartenberg (DH) parameters. The complete systematic method of defining the DH parameters will not be discussed here. The interested reader is referred to texts such as Vepa [1], where the application of the DH convention to robot manipulators is discussed in some detail.