

2D and 3D

Image Analysis by Moments

Jan Flusser, Tomáš Suk, and Barbara Zitová

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Prague, Czech Republic*

Presents recent significant and rapid developments in the field of 2D and 3D image analysis

2D and 3D Image Analysis by Moments, is a unique compendium of moment-based image analysis, which includes traditional methods and also reflects the latest developments of the field.

The book presents a survey of 2D and 3D moment invariants with respect to similarity and affine spatial transformations and to image blurring and smoothing by various filters. The book comprehensively describes the mathematical background and theorems about the invariants, but a large part is also devoted to practical usage of moments. Applications from various fields of computer vision, remote sensing, medical imaging, image retrieval, watermarking, and forensic analysis are demonstrated. Attention is also paid to efficient algorithms of moment computation.

Key features

- Presents a systematic overview of moment-based features used in 2D and 3D image analysis.
- Demonstrates invariant properties of moments with respect to various spatial and intensity transformations.
- Reviews and compares several orthogonal polynomials and respective moments.
- Describes efficient numerical algorithms for moment computation.
- Is a "classroom ready" textbook with a self-contained introduction to classifier design.
- Accompanying website contains around 300 lecture slides, Matlab codes, complete lists of the invariants, test images, and other supplementary material.

2D and 3D Image Analysis by Moments, is ideal for mathematicians, computer scientists, engineers, software developers, and Ph.D students involved in image analysis and recognition. With the addition of two introductory chapters on classifier design, the book may also serve as a self-contained textbook for graduate university courses on object recognition.



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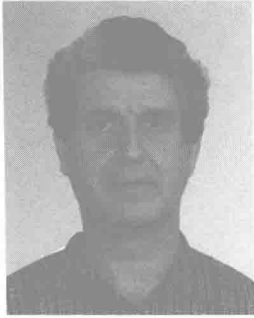
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In 2007 Jan Flusser received the Award of the Chairman of the Czech Science Foundation for the best research project and won the Prize of the Czech Academy of Sciences for his contribution to image fusion theory. In 2010, he was awarded the SCOPUS 1000 Award presented by Elsevier. He received the Felber Medal of the Czech Technical University for excellent contribution to research and education in 2015. Personal webpage: <http://www.utia.cas.cz/people/flusser>



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Tomáš Suk has authored and coauthored more than thirty journal papers and fifty conference papers in these areas, including tutorials on moment invariants held at the conferences ICIP'07 and SPPRA'09. He also coauthored the monograph *Moments and Moment Invariants in Pattern Recognition* (Wiley, 2009). His publications have received about approximately citations. In 2002 he received the Otto Wichterle Premium of the Czech Academy of Sciences for young scientists. Personal webpage: <http://zoi.utia.cas.cz/suk>



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Preface

Seven years ago we published our first monograph on moments and their applications in image recognition: J. Flusser, T. Suk, and B. Zitová, *Moments and Moment Invariants in Pattern Recognition*, Wiley, 2009.

That book (referred to as MMIPR) was motivated by the need for a monograph covering theoretical aspects of moments and moment invariants and their relationship to practical image recognition problems. Long before 2009, object recognition had become an established discipline inside image analysis. Moments and moment invariants, introduced to the image analysis community in the early 1960s, have played a very important role as features in invariant recognition. Nevertheless, such a book had not been available before 2009¹.

The development of moment invariants after 2009 was even more rapid than before. In SCOPUS, which is probably the most widely-used publication database, we have received 16,000 search results as the response to the “image moment” keyword and 6,000 results of the “moment invariants” search². There has been an overlap of about 2,000 papers, which results in 20,000 relevant papers in total. This huge number of entries illustrates how a large and important area of computer science has been formed by the methods based on image moments. In Figure 1 we can observe the development in time. A relatively slow growth in the last century was followed by a rapid increase of the number of publications in 2009–2010 (we believe that the appearance of MMIPR at least slightly contributed to the growing interest in moment invariants). Since then, the annual number of publications has slightly fluctuated, reaching another local maximum in 2014. In 2014, a new multi-authored book edited by G. A. Papakostas³ appeared on the market. Although the editor did a very good job, this book suffers from a common weakness of multi-authored books – the topics of individual chapters had been selected by their authors according to their interest, which made some areas overstressed, while some others, remained unmentioned despite their importance. The Papakostas book reflects recent developments in some areas but can hardly be used as a course textbook.

¹ The very first moment-focused book by R. Mukundan and K. R. Ramakrishnan, *Moment Functions in Image Analysis*, World Scientific, 1998, is just a short introduction to this field. The second book by M. Pawlak, *Image Analysis by Moments: Reconstruction and Computational Aspects*, Wrocław, Poland, 2006, is focused narrowly on numerical aspects of image moments, without providing a broader context of invariant image recognition and of practical applications.

² The search was performed within the title and abstract of the papers.

³ G. A. Papakostas ed., *Moments and Moment Invariants – Theory and Applications*, Science Gate Publishing, 2014.

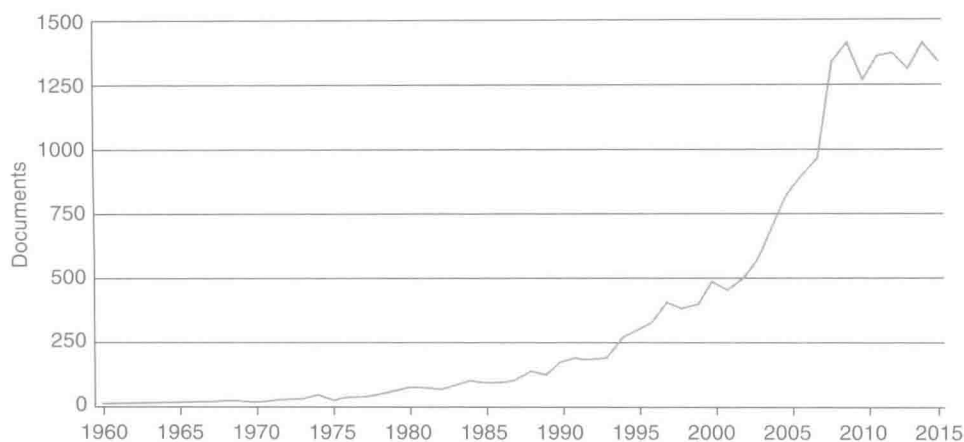


Figure 1 The number of moment-related publications as found in SCOPUS.

The great number of publications that have appeared since 2009 led us to the idea of writing a new, more comprehensive book on this topic. The abundant positive feedback we have received from the MMIPR readers and from our students was another factor which has strengthened our intentions. In 2014, the MMIPR was even translated into Chinese⁴.

The main goal of the book you are now holding in your hands is to be a comprehensive monograph covering the current state of the art of moment-based image analysis and presenting the latest developments in this field in a consistent form. In this book, the reader will find a survey of all important theoretical results as well as a description of how to use them in practical image analysis tasks. In particular, our aims were

- To review the development of moments and moment invariants after 2009;
- To cover the area of 3D moments and invariants, which were mostly omitted in the MMIPR but have become important in the last few years;
- To present some of the latest unpublished results, especially in the field of blur invariants; and
- To provide readers with an introduction to a broader context, showing the role of the moments in image analysis and recognition and reviewing other relevant areas and approaches.

At the same time, we aimed to write a self-contained book. This led us to the decision (with the kind permission of Wiley) to include also the core parts of the MMIPR, of course in enhanced and extended/up-to-date form. Attentive readers may realize that about one half of this book was already treated in the MMIPR in some form, while the other half is original. In particular, Chapters 1, 2, 4, and 6 are completely or mostly original. Chapters 7, 8, 9, and 10 are substantially extended versions of their ancestors in the MMIPR, and Chapters 3 and 5 were adopted from the MMIPR after minor updates.

⁴ Published by John Wiley & Univ. of Science and Technology of China Press, see <http://library.utia.cas.cz/separaty/2015/ZOI/flusser-0444327-cover.jpg>.

The book is based on our deep experience with moments and moment invariants gained from twenty-five years of research in this area, from teaching graduate courses on moment invariants and related fields at the Czech Technical University and at the Charles University, Prague, Czech Republic, and from presenting several tutorials on moments at major international conferences.

The target readership includes academic researchers and R&D engineers from all application areas who need to recognize 2D and 3D objects extracted from binary/graylevel/color images and who look for invariant and robust object descriptors, as well as specialists in moment-based image analysis interested in a new development on this field. Last but not least, the book is also intended for university lecturers and graduate students of image analysis and pattern recognition. It can be used as textbook for advanced graduate courses on Invariant Pattern Recognition. The first two chapters can be even utilized as supplementary reading to undergraduate courses on Pattern Recognition and Image Analysis.

We created an accompanying website at http://zoi.utia.cas.cz/moment_invariants2 containing selected Matlab codes, the complete lists of the invariants, the slides for those who wish to use this book for educational purposes, and Errata (if any). This website is free for the book readers (the password can be found in the book) and is going to be regularly updated.

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First of all, we would like to express gratitude to our employer, the Institute of Information Theory and Automation (ÚTIA) and to its mother organization, the Czech Academy of Sciences, for creating an inspiring and friendly environment and for continuous support of our research. We also thank both universities where we teach, the Czech Technical University, Faculty of Nuclear Science and Physical Engineering, and the Charles University, Faculty of Mathematics and Physics, for their support and for administration of all our courses.

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*Institute of Information Theory and Automation of the CAS,
Prague, Czech Republic, November 2015.*

About the companion website

Don't forget to visit the companion website for this book:

<http://www.wiley.com/go/flusser/2d3dimageanalysis>



There you will find valuable material designed to enhance your learning, including:

- 300 slides
- Matlab codes

Scan this QR code to visit the companion website

Contents

Preface	xvii
Acknowledgements	xxi
1 Motivation	1
1.1 Image analysis by computers	1
1.2 Humans, computers, and object recognition	4
1.3 Outline of the book	5
References	7
2 Introduction to Object Recognition	8
2.1 Feature space	8
2.1.1 <i>Metric spaces and norms</i>	9
2.1.2 <i>Equivalence and partition</i>	11
2.1.3 <i>Invariants</i>	12
2.1.4 <i>Covariants</i>	14
2.1.5 <i>Invariant-less approaches</i>	15
2.2 Categories of the invariants	15
2.2.1 <i>Simple shape features</i>	16
2.2.2 <i>Complete visual features</i>	18
2.2.3 <i>Transformation coefficient features</i>	20
2.2.4 <i>Textural features</i>	21
2.2.5 <i>Wavelet-based features</i>	23
2.2.6 <i>Differential invariants</i>	24
2.2.7 <i>Point set invariants</i>	25
2.2.8 <i>Moment invariants</i>	26
2.3 Classifiers	27
2.3.1 <i>Nearest-neighbor classifiers</i>	28
2.3.2 <i>Support vector machines</i>	31
2.3.3 <i>Neural network classifiers</i>	32
2.3.4 <i>Bayesian classifier</i>	34
2.3.5 <i>Decision trees</i>	35
2.3.6 <i>Unsupervised classification</i>	36
2.4 Performance of the classifiers	37

2.4.1	<i>Measuring the classifier performance</i>	37
2.4.2	<i>Fusing classifiers</i>	38
2.4.3	<i>Reduction of the feature space dimensionality</i>	38
2.5	Conclusion	40
	References	41
3	2D Moment Invariants to Translation, Rotation, and Scaling	45
3.1	Introduction	45
3.1.1	<i>Mathematical preliminaries</i>	45
3.1.2	<i>Moments</i>	47
3.1.3	<i>Geometric moments in 2D</i>	48
3.1.4	<i>Other moments</i>	49
3.2	TRS invariants from geometric moments	50
3.2.1	<i>Invariants to translation</i>	50
3.2.2	<i>Invariants to uniform scaling</i>	51
3.2.3	<i>Invariants to non-uniform scaling</i>	52
3.2.4	<i>Traditional invariants to rotation</i>	54
3.3	Rotation invariants using circular moments	56
3.4	Rotation invariants from complex moments	57
3.4.1	<i>Complex moments</i>	57
3.4.2	<i>Construction of rotation invariants</i>	58
3.4.3	<i>Construction of the basis</i>	59
3.4.4	<i>Basis of the invariants of the second and third orders</i>	62
3.4.5	<i>Relationship to the Hu invariants</i>	63
3.5	Pseudoinvariants	67
3.6	Combined invariants to TRS and contrast stretching	68
3.7	Rotation invariants for recognition of symmetric objects	69
3.7.1	<i>Logo recognition</i>	75
3.7.2	<i>Recognition of shapes with different fold numbers</i>	75
3.7.3	<i>Experiment with a baby toy</i>	77
3.8	Rotation invariants via image normalization	81
3.9	Moment invariants of vector fields	86
3.10	Conclusion	92
	References	92
4	3D Moment Invariants to Translation, Rotation, and Scaling	95
4.1	Introduction	95
4.2	Mathematical description of the 3D rotation	98
4.3	Translation and scaling invariance of 3D geometric moments	100
4.4	3D rotation invariants by means of tensors	101
4.4.1	<i>Tensors</i>	101
4.4.2	<i>Rotation invariants</i>	102
4.4.3	<i>Graph representation of the invariants</i>	103
4.4.4	<i>The number of the independent invariants</i>	104
4.4.5	<i>Possible dependencies among the invariants</i>	105
4.4.6	<i>Automatic generation of the invariants by the tensor method</i>	106

4.5	Rotation invariants from 3D complex moments	108
4.5.1	<i>Translation and scaling invariance of 3D complex moments</i>	112
4.5.2	<i>Invariants to rotation by means of the group representation theory</i>	112
4.5.3	<i>Construction of the rotation invariants</i>	115
4.5.4	<i>Automated generation of the invariants</i>	117
4.5.5	<i>Elimination of the reducible invariants</i>	118
4.5.6	<i>The irreducible invariants</i>	118
4.6	3D translation, rotation, and scale invariants via normalization	119
4.6.1	<i>Rotation normalization by geometric moments</i>	120
4.6.2	<i>Rotation normalization by complex moments</i>	123
4.7	Invariants of symmetric objects	124
4.7.1	<i>Rotation and reflection symmetry in 3D</i>	124
4.7.2	<i>The influence of symmetry on 3D complex moments</i>	128
4.7.3	<i>Dependencies among the invariants due to symmetry</i>	130
4.8	Invariants of 3D vector fields	131
4.9	Numerical experiments	131
4.9.1	<i>Implementation details</i>	131
4.9.2	<i>Experiment with archeological findings</i>	133
4.9.3	<i>Recognition of generic classes</i>	135
4.9.4	<i>Submarine recognition – robustness to noise test</i>	137
4.9.5	<i>Teddy bears – the experiment on real data</i>	141
4.9.6	<i>Artificial symmetric bodies</i>	142
4.9.7	<i>Symmetric objects from the Princeton Shape Benchmark</i>	143
4.10	Conclusion	147
	Appendix 4.A	148
	Appendix 4.B	156
	Appendix 4.C	158
	References	160
5	Affine Moment Invariants in 2D and 3D	163
5.1	Introduction	163
5.1.1	<i>2D projective imaging of 3D world</i>	164
5.1.2	<i>Projective moment invariants</i>	165
5.1.3	<i>Affine transformation</i>	167
5.1.4	<i>2D Affine moment invariants – the history</i>	168
5.2	AMIs derived from the Fundamental theorem	170
5.3	AMIs generated by graphs	171
5.3.1	<i>The basic concept</i>	172
5.3.2	<i>Representing the AMIs by graphs</i>	173
5.3.3	<i>Automatic generation of the invariants by the graph method</i>	173
5.3.4	<i>Independence of the AMIs</i>	174
5.3.5	<i>The AMIs and tensors</i>	180
5.4	AMIs via image normalization	181
5.4.1	<i>Decomposition of the affine transformation</i>	182
5.4.2	<i>Relation between the normalized moments and the AMIs</i>	185
5.4.3	<i>Violation of stability</i>	186

5.4.4	<i>Affine invariants via half normalization</i>	187
5.4.5	<i>Affine invariants from complex moments</i>	187
5.5	The method of the transvectants	190
5.6	Derivation of the AMIs from the Cayley-Aronhold equation	195
5.6.1	<i>Manual solution</i>	195
5.6.2	<i>Automatic solution</i>	198
5.7	Numerical experiments	201
5.7.1	<i>Invariance and robustness of the AMIs</i>	201
5.7.2	<i>Digit recognition</i>	201
5.7.3	<i>Recognition of symmetric patterns</i>	204
5.7.4	<i>The children's mosaic</i>	208
5.7.5	<i>Scrabble tiles recognition</i>	210
5.8	Affine invariants of color images	214
5.8.1	<i>Recognition of color pictures</i>	217
5.9	Affine invariants of 2D vector fields	218
5.10	3D affine moment invariants	221
5.10.1	<i>The method of geometric primitives</i>	222
5.10.2	<i>Normalized moments in 3D</i>	224
5.10.3	<i>Cayley-Aronhold equation in 3D</i>	225
5.11	Beyond invariants	225
5.11.1	<i>Invariant distance measure between images</i>	225
5.11.2	<i>Moment matching</i>	227
5.11.3	<i>Object recognition as a minimization problem</i>	229
5.11.4	<i>Numerical experiments</i>	229
5.12	Conclusion	231
	Appendix 5.A	232
	Appendix 5.B	233
	References	234
6	Invariants to Image Blurring	237
6.1	Introduction	237
6.1.1	<i>Image blurring – the sources and modeling</i>	237
6.1.2	<i>The need for blur invariants</i>	239
6.1.3	<i>State of the art of blur invariants</i>	239
6.1.4	<i>The chapter outline</i>	246
6.2	An intuitive approach to blur invariants	247
6.3	Projection operators and blur invariants in Fourier domain	249
6.4	Blur invariants from image moments	252
6.5	Invariants to centrosymmetric blur	254
6.6	Invariants to circular blur	256
6.7	Invariants to N-FRS blur	259
6.8	Invariants to dihedral blur	265
6.9	Invariants to directional blur	269
6.10	Invariants to Gaussian blur	272
6.10.1	<i>1D Gaussian blur invariants</i>	274
6.10.2	<i>Multidimensional Gaussian blur invariants</i>	278

6.10.3	<i>2D Gaussian blur invariants from complex moments</i>	279
6.11	Invariants to other blurs	280
6.12	Combined invariants to blur and spatial transformations	282
6.12.1	<i>Invariants to blur and rotation</i>	282
6.12.2	<i>Invariants to blur and affine transformation</i>	283
6.13	Computational issues	284
6.14	Experiments with blur invariants	285
6.14.1	<i>A simple test of blur invariance property</i>	285
6.14.2	<i>Template matching in satellite images</i>	286
6.14.3	<i>Template matching in outdoor images</i>	291
6.14.4	<i>Template matching in astronomical images</i>	291
6.14.5	<i>Face recognition on blurred and noisy photographs</i>	292
6.14.6	<i>Traffic sign recognition</i>	294
6.15	Conclusion	302
	Appendix 6.A	303
	Appendix 6.B	304
	Appendix 6.C	306
	Appendix 6.D	308
	Appendix 6.E	310
	Appendix 6.F	310
	Appendix 6.G	311
	References	315
7	2D and 3D Orthogonal Moments	320
7.1	Introduction	320
7.2	2D moments orthogonal on a square	322
7.2.1	<i>Hypergeometric functions</i>	323
7.2.2	<i>Legendre moments</i>	324
7.2.3	<i>Chebyshev moments</i>	327
7.2.4	<i>Gaussian-Hermite moments</i>	331
7.2.5	<i>Other moments orthogonal on a square</i>	334
7.2.6	<i>Orthogonal moments of a discrete variable</i>	338
7.2.7	<i>Rotation invariants from moments orthogonal on a square</i>	348
7.3	2D moments orthogonal on a disk	351
7.3.1	<i>Zernike and Pseudo-Zernike moments</i>	352
7.3.2	<i>Fourier-Mellin moments</i>	358
7.3.3	<i>Other moments orthogonal on a disk</i>	361
7.4	Object recognition by Zernike moments	363
7.5	Image reconstruction from moments	365
7.5.1	<i>Reconstruction by direct calculation</i>	367
7.5.2	<i>Reconstruction in the Fourier domain</i>	369
7.5.3	<i>Reconstruction from orthogonal moments</i>	370
7.5.4	<i>Reconstruction from noisy data</i>	373
7.5.5	<i>Numerical experiments with a reconstruction from OG moments</i>	373
7.6	3D orthogonal moments	377
7.6.1	<i>3D moments orthogonal on a cube</i>	380

7.6.2	<i>3D moments orthogonal on a sphere</i>	381
7.6.3	<i>3D moments orthogonal on a cylinder</i>	383
7.6.4	<i>Object recognition of 3D objects by orthogonal moments</i>	383
7.6.5	<i>Object reconstruction from 3D moments</i>	387
7.7	Conclusion	389
	References	389
8	Algorithms for Moment Computation	398
8.1	Introduction	398
8.2	Digital image and its moments	399
8.2.1	<i>Digital image</i>	399
8.2.2	<i>Discrete moments</i>	400
8.3	Moments of binary images	402
8.3.1	<i>Moments of a rectangle</i>	402
8.3.2	<i>Moments of a general-shaped binary object</i>	403
8.4	Boundary-based methods for binary images	404
8.4.1	<i>The methods based on Green's theorem</i>	404
8.4.2	<i>The methods based on boundary approximations</i>	406
8.4.3	<i>Boundary-based methods for 3D objects</i>	407
8.5	Decomposition methods for binary images	410
8.5.1	<i>The "delta" method</i>	412
8.5.2	<i>Quadtree decomposition</i>	413
8.5.3	<i>Morphological decomposition</i>	415
8.5.4	<i>Graph-based decomposition</i>	416
8.5.5	<i>Computing binary OG moments by means of decomposition methods</i>	420
8.5.6	<i>Experimental comparison of decomposition methods</i>	422
8.5.7	<i>3D decomposition methods</i>	423
8.6	Geometric moments of graylevel images	428
8.6.1	<i>Intensity slicing</i>	429
8.6.2	<i>Bit slicing</i>	430
8.6.3	<i>Approximation methods</i>	433
8.7	Orthogonal moments of graylevel images	435
8.7.1	<i>Recurrent relations for moments orthogonal on a square</i>	435
8.7.2	<i>Recurrent relations for moments orthogonal on a disk</i>	436
8.7.3	<i>Other methods</i>	438
8.8	Conclusion	440
	Appendix 8.A	441
	References	443
9	Applications	448
9.1	Introduction	448
9.2	Image understanding	448
9.2.1	<i>Recognition of animals</i>	449
9.2.2	<i>Face and other human parts recognition</i>	450
9.2.3	<i>Character and logo recognition</i>	453
9.2.4	<i>Recognition of vegetation and of microscopic natural structures</i>	454