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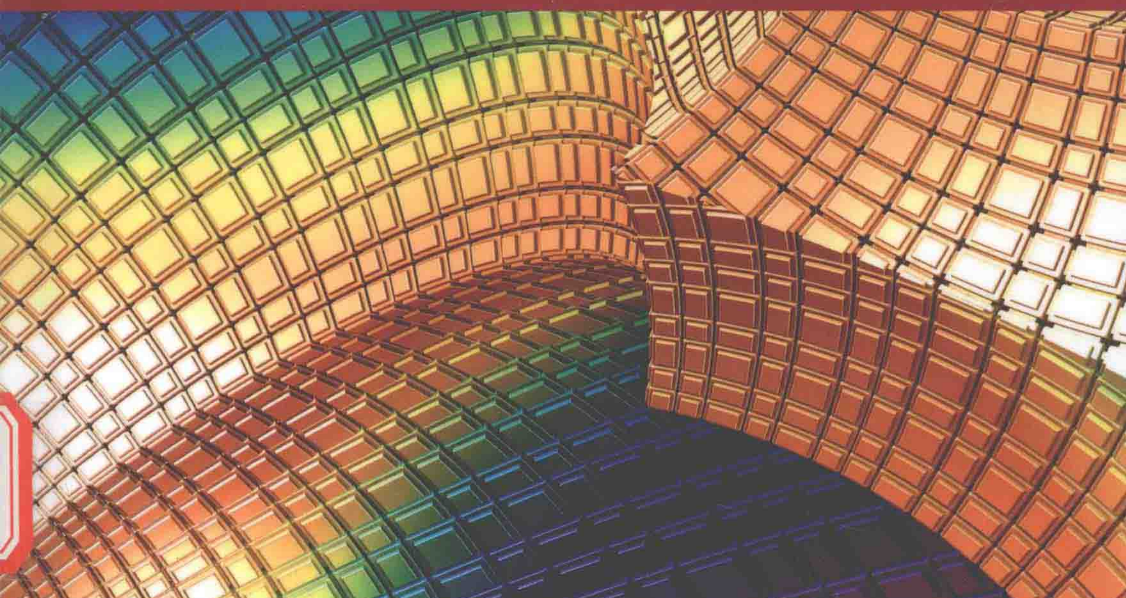
Galilean Mechanics and Thermodynamics of Continua

Géry de Saxcé and Claude Vallée



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First published 2016 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2015957652

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-84821-642-6

Foreword

In this very well-written book, my colleagues and friends Géry de Saxcé and Claude Vallée present a general framework in which the laws of mechanics are formulated, which govern equilibria and motions of material bodies in space. They show that this framework is also convenient for the formulation of the laws of thermodynamics which involve, besides the notions of mass, velocity and acceleration already in use in mechanics, the notions of heat, entropy and temperature. To allow the readers to fully appreciate the originality and the interest of their work, I am going to briefly remind them of the evolution, from the 17th Century up to our time, of the ideas of scientists about the physical space and time and about material bodies' motions.

According to Isaac Newton's (1642–1727) ideas expressed in his famous book[1], physical time and space are two separate entities, both having an absolute character. Physical time can be mathematically represented by a straight line, spreading to infinity in both directions, endowed with a geometric structure which allows the comparison of the lengths of two time intervals, even when they are separated by several centuries of millenia. Consequently, the choice of a unit of time and an arbitrary date as an origin establishes a one-to-one correspondence between elements of the physical time and elements of the real line $\mathbb{R}$. As for the physical space, Newton identifies it with the three-dimensional space of Euclidean geometry; the choice of a unit of length allows its identification with (in modern mathematical language) a three-dimensional Euclidean affine space, in which we can measure distances and angles, and apply all theorems of Euclid's geometry. Each material body occupies, during its existence, a certain position in space, which may depend on time. It is at rest if the positions of all its material elements remain fixed along the time, and in motion in the opposite case. Its motion is described by all the curves,

1 Isaac Newton, *Philosophiae naturalis principia mathematica*, 1687.

drawn in the physical space and parametrized by the time, made by the successive positions of each of its material elements.

Newton was well aware of the fact that the position and the motion of a material body in the physical space are always appreciated relatively to the positions of other physical bodies used to determine a reference frame. He formulated the fundamental law of dynamics (which states that the acceleration of a punctual material body is equal to the quotient by its mass of the force which acts on it) for the absolute motion of a material body in the physical space. But immediately, he noticed that this fundamental law remains valid for the *relative motion* of the material body with respect to a reference frame whose absolute motion is a motion by translations at a constant velocity.

Newton's ideas about time, space and the mathematical description of motions of material bodies were soon criticized, notably by Gottfried Wilhelm Leibniz (1646–1716), who believed that the concept of an absolute space was useless, had no real existence and that the laws governing material bodies' motions should be formulated in a way involving only the relative position of each body with respect to all the other bodies. Unfortunately the mathematical concepts needed to translate Leibniz' ideas into a usable theory were not available at his time. Much later, Ernst Mach (1838–1916)[2], who thought that the inertia of a material body was due to the actions on it of all other material bodies present in the universe, criticized Newton's ideas about the absolute character of space and time, as well as the principles of inertia and equality of action and reaction. Mach's ideas could not be incorporated into a usable theory in mechanics, but they influenced Albert Einstein when he developed the theory of general relativity.

In spite of these criticisms, Newton's ideas about space, time and motion are, essentially, still in use nowadays in classical mechanics. Of course, the progress of astronomy has shown that nothing is at rest in the universe, leading many scientists to become doubtful about the existence of an absolute space. But they found a way to avoid the use of that concept: when Newton's laws of dynamics can be applied to the *relative motions* of material bodies with respect to some reference frame, that frame was said to be *inertial*, or *Galilean*. It is then easy to show that when a given reference frame R_1 is inertial, another reference frame R_2 is also inertial if and only if it moves relatively with respect to R_1 by translations at a constant velocity. Instead of assuming the existence of an absolute space, it is enough to assume the existence of *one* inertial reference frame, which implies the existence of *an infinite number* of such frames, each in relative motion by translations at a constant velocity with

2 Ernst Mach, *Die Mechanik in ihrer Entwicklung, Historisch-kritisch dargestellt*, 1883. First English translation by T. J. McCormack, under the title *The Science of Mechanics*, Chicago, 1893.

respect to each other. Mechanicians became accustomed to only considering relative motions of material bodies with respect to (preferentially approximately inertial) reference frames, and to use concepts (such as velocity and kinetic energy) which depend on the considered body and also on the reference frame with respect to which its motion is studied. The use of *fictitious forces* (centrifugal and Coriolis' forces) even made possible the use of non-inertial frames and the study, for example, of the relative motion of the *Foucault's pendulum* with respect to the Earth's reference frame (discussed in Chapter 3). By preventing any interrogation about the mathematical tools used to represent the physical space and time, the immoderate use of reference frames seems to have delayed the discovery of the special theory of relativity. I am going to recall the main steps of this discovery. Then, I will speak about that of the general theory of relativity.

At the beginning of the decade 1860–1869, James Clerk Maxwell (1831–1879) established the equations governing electromagnetic phenomena and introduced the concept of *field*. According to these equations, perturbations of an electromagnetic field propagate as waves at a finite velocity, which does not depend on the motion of the source of the perturbations and is the same in all directions of propagation. Observing that the numerical value of that velocity was close to that of light's velocity, Maxwell understood that light is an electromagnetic wave. In classical kinematics, a phenomenon can propagate at the same velocity in all directions only with respect to a particular reference frame. Physicists, who no more really believed in Newton's absolute space, then assumed the existence of a very subtle medium, filling empty space and impregnating all material bodies, in which the propagation of electromagnetic waves occurred. They called *luminiferous ether* that hypothetic medium. They thought that it was in a reference frame with respect to which the luminiferous ether is at rest that the relative velocity of light was the same in all directions. Under this assumption, careful measurements of the velocity of light in various directions, made at different dates at which the Earth's velocity on its orbit around the Sun takes different values, could detect the relative velocity of the Earth with respect to the luminiferous ether. These measurements were made around 1887 by Albert Abraham Michelson (1852–1931) and Edward William Morley (1838–1923). No perceptible velocity of the Earth with respect to the luminiferous ether could be detected. No satisfying explanation of this result was found until 1905.

Albert Einstein (1879–1955) offered[3], in 1905, a truly revolutionary explanation. He clearly understood that light's property to propagate at the same velocity in all directions, whatever the reference frame with respect to which its relative velocity is evaluated, is incompatible with the absolute character of the notion of simultaneity of

3 Albert Einstein, *Zur Electrodynamic bewegter Körper*, translated by M. Saha under the title *On the Electrodynamics of Moving Bodies*, Calcutta, 1920.

two events occurring at two different places in space. He proposed as a new principle, called the *Principle of Relativity*, the fact that all inertial reference frames are equivalent, for electromagnetic phenomena as well as for mechanical phenomena. He proposed as a second principle the fact that the light propagates at the same velocity in all directions, independently of the motion of its source and the inertial reference frame with respect to which that velocity is evaluated. On these two founding principles, giving up the absolute character of simultaneity of two events occurring at different places, therefore the concept of an absolute time, he succeed in building a coherent theory. The principle of relativity led him to give up the notions of absolute rest and absolute motion, and he clearly saw that in his new theory the concept of luminiferous ether was no more useful.

In the same year, Jules Henri Poincaré (1854–1912) published a Note in the *Comptes rendus de l'Académie* and a much longer paper[4] in which he introduced "local times" at which an event occurs, depending on that event *and on the frame in which that event is perceived by an observer* which, together with the three space coordinates of the place at which that event occurs, make a system of four coordinates of the event in space-time. He studied the transformation laws, which he called *Lorentz transformations* in honor of Hendrik Anton Lorentz (1853–1928), which give the four space-time coordinates of an event in some inertial reference frame as functions of the four space-time coordinates of *the same event* in *another inertial reference frame*. Lorentz transformations can also be seen as the transformation laws which give the four space-time coordinates of an event in some inertial reference frame as functions of the four space-time coordinates of *another event* in *the same inertial reference frame*. Poincaré proved that the set of all Lorentz transformations is a group and determined its invariants. He saw that the local times of the same event seen by two different observers are different, since he determined the formula which links these two local times. He also saw that Lorentz transformations are in agreement with the fact that light propagates at the same velocity in all directions, which does not depend on the reference frame with respect to which that velocity is evaluated, since the light velocity appears as an invariant of the group of all Lorentz transformations. But, he did not state as clearly as it was stated by Einstein the fact that the concept of an absolute time should be discarded, nor the fact that the concept of luminiferous ether is useless.

4 Henri Poincaré, *La Mécanique nouvelle*, the book brings together in a single volume the text of a conference in Lille of the Association française pour l'avancement des sciences in 1909, the note of 23 July 1905 entitled *Sur la dynamique de l'électron*, published by Rendiconti del Circolo matematico di Palermo **XXI** (1906) and a note by the Académie des Sciences, of the same title (15 June 1905, **CXL**, 1905, p. 1504); Gauthier-Villars, Paris, 1924; reprinted by Éditions Jacques Gabay, Paris, 1989.

Hermann Minkowski (1864–1909) precisely described, in 1908[5], the geometric structure of space-time in the special theory of relativity, which today bears his name: it is a four-dimensional affine space endowed with a pseudo-Euclidean scalar product of signature $(+,-,-,-)$. Lorentz transformations are linear automorphisms of the associated pseudo-Euclidean vector space which leave invariant that scalar product. Without giving its formal definition, Poincaré already considered this space-time and studied its geometric properties in his papers published in 1905. For this reason the group of affine transformations of the Minkowski space-time which preserve its structure today is called *Poincaré's group*.

As in classical mechanics, there exists in the theory of special relativity privileged reference frames: the inertial frames. These reference frames are global: each of them totally includes space and time. Einstein wanted to build a theory only using *local* reference frames, none of them being privileged. He also noticed that the special theory of relativity does not explain a disturbing fact: the equality of inertial and gravitational masses. This equality implies (and is equivalent to) the identity of nature between acceleration fields and gravitational fields: in a reference frame suitably accelerated, it is possible either to annihilate, or to create a gravitational field in a limited part of space-time. Einstein proposed to consider this fact as a principle, and called it the *equivalence principle*. To account for this principle, he had the brilliant idea of including gravitational fields into the geometric properties of space-time. The mass–energy equivalence, which he discovered while developing the special theory of relativity, led him to think that not only mass, but all forms of energy (for example, electromagnetic energy) must contribute to the creation of a gravitational field. With these ideas, he built a coherent mathematical theory, which he called the *general theory of relativity*, and published it in four successive papers in 1915[6].

In the general theory of relativity, space-time is no longer an affine space, as it is in classical mechanics and in the special theory of relativity: it is a four-dimensional differential manifold endowed, once a unit of time (or, equivalently, of length) is chosen, with a pseudo-Riemannian metric of signature $(+,-,-,-)$. It is no longer a frame with fixed geometric properties in which physical phenomena occur. In

5 H. Minkowski, Talk presented in Cologne on the 21th september 1908, published in the book by H. A. Lorentz, A. Einstein and H. Minkowski *Das Relativitätsprinzip; eine Sammlung von Abhandlungen*, B. G. Teubner, Leipzig, Berlin 1922. Analyzed in the book by René Dugas *Histoire de la Mécanique*, Éditions du Griffon, Neuchâtel, 1950, reprinted by Éditions Jacques Gabay, Paris, 1996, pp. 468–473.

6 Albert Einstein, *Fundamental Ideas of the General Theory of Relativity and the Application of this Theory in Astronomy*, *On the General Theory of Relativity*, *Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity*, *The Field Equations of Gravitation*, Preussische Akademie der Wissenschaften, Sitzungsberichte, 1915 part 1 p. 315, part 2 pp. 778–786, 799–801, 831–839, 844–847.

Maxwell's theory of electromagnetism, an electromagnetic field in which an electrically charged particle is moving acts on the motion of that particle. Conversely, the motion of that electrically charged particle creates an electromagnetic field, therefore modifies the field in which its motion takes place, and acts on the motions of other electrically charged particles. A similar reciprocity exists in the theory of general relativity: the mass of a material body, and more generally any kind of energy, is acted upon by the gravitational field (included in the geometry of space-time) and that action affects its motion; conversely, that mass or energy participates in the creation of the gravitational field, therefore acts on the geometry of space-time. The notion of a straight line on which a particle moves at a constant velocity is no longer valid in general relativity: the world line of a material particle acted upon by gravitation only is a time-like geodesic of the Levi-Civita connection associated with the pseudo-Riemannian metric of space-time. There is no perfectly rigid body, nor instantaneous action of a body on another non-coincident body in general relativity space-time. Actions at a distance only occur by fields, which propagate at a velocity not exceeding the velocity of light, whose evolution must be determined together with the motions of material bodies on which they act. It makes rather cumbersome the practical use of general relativity theory, which explains why classical, non-relativistic mechanics keeps its usefulness when the mechanical phenomena under consideration only involve material bodies each of whose relative velocity with respect to the other bodies is very small compared to the velocity of light.

The great French geometer Élie Cartan (1869–1951) very soon understood that the mathematical tools and the ideas of general relativity, especially the concept of space-time, could be used advantageously in classical mechanics. In a two-part paper published in 1923 and 1924[7], he introduced and investigated the notion of an *affine connection* on a differential manifold and explained how that notion could be used to include gravitational forces in the geometry of space-time in the framework of classical mechanics. The notion of an affine connection has its sources in the works of Tullio Levi-Civita (1873–1941), Hermann Weyl (1885–1955) and Élie Cartan himself. It was generalized and made clearer by Charles Ehresmann (1905–1979). A smooth path in a differential manifold being given, an affine connection allows us to define the parallel transport of an affine frame of the tangent space at a point of that path toward the tangent spaces at all other points of the path. It offers, therefore, a way to identify all the spaces tangent at various points of the path to a single affine space, which can be used as a local model of the manifold in a neighborhood of the path. From 1929 until 1932, Élie Cartan had a regular correspondence with Albert

7 Élie Cartan, *Les variétés à connexion affine et la théorie de la relativité généralisée*, I et II, Ann. Ec. Norm. 40, 1923, pp. 325–342 and 41, 1924, pp. 1–25. Ces articles se trouvent aussi dans ses *œuvres complètes*, partie III 1, pp. 659–746 and 799–823. Éditions du CNRS, Paris, 1984.

Einstein[8] about the concept of parallel transport, which the latter wanted to use in a new theory in which electromagnetic fields would be included in the geometry of space-time in a way similar to that in which he included gravitational fields in that geometry.

In the four-dimensional space-time of classical mechanics considered by Élie Cartan, the time keeps an absolute character: to each element (called "event") in space-time corresponds a well-defined element of the time, the instant when that event occurs. The set at all events which occur at a given instant is a three-dimensional submanifold of space-time, called the *space at that instant*. By assuming that the space at each given instant is (once chosen an unit of length) an affine Euclidean three-dimensional space, we make valid the notion of a perfectly rigid solid body and usable all the theorems of Euclid's geometry, exactly as they are in usual classical mechanics. Instantaneous actions at a distance can also be considered in that space-time, which for these reasons is better suited for the treatment of problems usually encountered in classical mechanics than the special or the general theories of relativity. A reference frame in that space-time is determined by a three-dimensional body R (which may be material or conceptual, as for example the set of three straight lines which join the Sun's center to three distant stars) which remains approximately rigid during some time interval I. For each pair (t_1, t_2) of instants in I, there exists a unique isometry of the space at t_1 onto the space at t_2 which maps the position of the body R at t_1 onto its position at t_2. By using these isometries to identify between themselves all these affine Euclidean spaces, we obtain an "abstract" three-dimensional affine Euclidean space in which the body R is at rest. The part of space-time made by all events which occur at an instant in I can, therefore, be identified with the product of that abstract three-dimensional Euclidean space with the time interval I. To study the relative motion of a mechanical system with respect to the reference frame determined by the body R amounts to use that identification. The reference frame determined by the body R is inertial if, with that identification, the motion of a material point which is not submitted to any force occurs on a straight line at a constant velocity. This is the *principle of inertia* discovered by Galileo Galilei (1564–1642), later included by Newton in his fundamental laws of mechanics.

With the exception of Jean-Marie Souriau and his coworkers, not many scientists working in classical mechanics granted much interest to the ideas of Élie Cartan about the use of space-time in their field of research. However, inconsistencies appearing in some formulations of constitutive laws governing large deformations of material bodies, Walter Noll (born in 1925) formulated his *principle of material objectivity*, which he renamed later, in agreement with his former scientific advisor Clifford Ambrose Truesdell (1919–2000) *principle of material frame indifference*.

8 Élie Cartan and Albert Einstein, *Letters on absolute parallelism 1929–1932*, Princeton University Press and Académie Royale de Belgique, Princeton, 1979.

On the webpage[9] at Carnegie Mellon University presenting his recent, still unpublished works, he gives the following formulation of this principle, which applies to any physical system: *the constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external frame of reference is used to describe them.* He then indicates several examples of application of this principle and writes: *"It is possible to make the principle of material frame indifference vacuously satisfied by using an intrinsic mathematical frame-work that does not use a frame-space at all when describing the internal interactions of a physical system.* The use of space-time, even in classical mechanics, is in my opinion a very good way to put this idea into practice. It is the approach used in this book by my colleague and friend Géry de Saxcé.

Charles-Michel MARLE
November 2015

9 http://www.math.cmu.edu/wn0g/

Introduction

General Relativity is not solely
a theory of gravitation which is reduced to the
prediction of tiny effects such as bending of light
or corrections to Mercurys orbital precession
but may be above all it is a consistent framework
for mechanics and physics of continua...

I.1. A geometrical viewpoint

"$A\gamma\varepsilon\omega\mu\varepsilon\tau\rho\eta\tau o\varsigma$ $\mu\eta\delta\varepsilon\iota\varsigma$ $\varepsilon\iota\sigma\iota\tau\omega$" ("Let none but geometers enter here"). According to the tradition, this phrase was inscribed above the entrance to Plato's academy. Because of the simplicity and beauty of its concepts, geometry was considered by Plato as essential preamble in training to acquire rigor. It is in this spirit that this book was written, setting the geometrical methods into the heart of the mechanics. This is precisely the philosophy of general relativity that is adopted here but restricted to the Galilean frame to describe phenomena for which the velocity of the light is so huge as it may be considered as infinite. This general point of view does not prevent allowing us occasionally short incursions into standard general relativity.

Mechanics is an experimental and theoretical science. Both of these aspects are indispensable. Even if this book is devoted to the modeling, we have to keep in mind that a mechanical theory makes sense only if its predictions agree with the experimental observations. Among the physical sciences, mechanics is certainly the oldest one and, precisely for this reason, it is the most mathematical one. It might also be said it is the most physical science among the mathematical ones. At the hinge between physics and mathematics, this book presents a new mathematical frame for continuum mechanics. In this sense, it may be considered as a part of applied mathematics but it also turns out to be what J.-J. Moreau called "Applied Mechanics to the Mathematics" in the sense that we revisit some pages of mathematics.

But why is mathematics needed to do mechanics? Of course, it is possible to do mechanics "with the hands" but mathematics is a language allowing us to describe the reality in a more accurate way. As J.-M. Souriau says in the "Grammaire de la Nature" [SOU 07]: "*Les chaussures sont un outil pour marcher; les mathématiques, un outil pour penser. On peut marcher sans chaussures, mais on va moins loin*" ("The shoes are a tool to walk; the mathematics, a tool to think. One can walk without shoes, but one goes less far").

I.2. Overview

Our aim is to present a unified approach of continuum mechanics not only for undergraduate, postgraduate and PhD students, but also for researchers and colleagues, without, however, being exhaustive. The sound ideas structuring mechanics are systematically emphasized and many topics are skimmed over, referring to technical works for more detailed developments. The presentation is progressive, inductive and bottom-up, from the basic subjects, at the Bachelor and Master degree levels, up to the most advanced topics and open questions, at the PhD degree level. Each degree level corresponds to part of the book, the latter providing a canvas for revisting the former two parts in which special comments and cross-references to the third part are indicated as "comments for experts". Useful mathematical definitions are recalled in the final chapter of each part.

I.2.1. *Part 1: particles and rigid bodies*

Except for Chapter 6, the first part corresponds to subjects taught at Bachelor degree level, needing only elementary mathematical tools of linear algebra, differential and integral calculus recalled in Chapter 7 at the end of the first part.

Chapter 1 is devoted to the modeling of the space-time of 4 dimensions and the principle of Galilean relativity. It is essential and must not be skipped. The Galilean transformations are coordinate changes preserving uniform straight motion, durations, distances and angles, and oriented volumes. The statements of the physical laws are postulated to be the same in all the coordinate systems deduced from each other by a Galilean transformation. This principle will be an Ariadne's thread all the way through this book.

The method used in the following four chapters is founded on a key object called a torsor which will be given for the continuous media of 1 and 5 dimensions. Chapter 2 deals with the statics of bodies. Introducing the force torsor, an object equipped with a force and a moment, we deduce the transport law of the moment in a natural way. Usual tools to study the equilibrium are the free body diagram, internal and external forces.

Chapter 3 is devoted to the dynamics of particles and gravitation. Tackling the dynamics is simply a matter of recovering an extra dimension, the time, leading to the dynamical torsor. The boost method reveals its components, the mass, the linear momentum, the passage and the angular momentum. After representing the rigid motions due to the Galilean coordinate systems, we model the Galilean gravitation, an object with two components, gravity and spinning, and we deduce the equation of motion. We state Newton's law of gravitation and solve the 2-body problem. We define the minimal properties expected from the other forces. As for application, we discuss Foucault's pendulum and model rocket thrust.

Chapter 4 applies the concepts of Chapter 2 to arches, slender bodies which, if they are seen from a long way off, can be considered as geometrically reduced to their mean line. Generalizing the methods developed previously, we obtain the local equilibrium equations of the arches and, using a frame moving along this line, a generalized corotational form of these equations. The concepts are illustrated by applications, a helical coil spring, a suspension bridge, a drilling riser and a cantilever beam.

Chapter 5 extends the tools developed in the previous chapters to study the dynamics of rigid bodies. The Lagrangian or material description is opposed to the Eulerian or spatial one. The body motion can be characterized by the co-torsor, an object equipped with a velocity and a spin. After introducing the mass-center, we construct the dynamical torsor and the kinetic energy of a body as extensive quantities. Next, we generalize the equation of motion to study the motion of the body around it. As for application, we present Poinsot's geometrical construction for free bodies and we deduce three integrals of the motion for a body with a contact point, i.e. Lagrange's top.

Chapter 6 is devoted to the calculus of variation which allows us to deduce from the minimum of a function, called the action, the equations of motion in a more abstract way than in Chapter 3. The principle of least action has over all a mnemonic value which allows deducing these laws in a consistent and systematic way. Such a principle presupposes that the Galilean gravitation is generated by a set of 4 potentials, not unique but defined *modulo* an arbitrary gauge function. We also introduce the Hamiltonian formalism and the canonical equations.

I.2.2. *Part 2: continuous media*

The second part corresponds to subjects taught at Master degree level, requiring more advanced mathematical tools of linear algebra and analysis such as partial derivative equations and tensorial calculus. In particular, if you are not familiar with the affine tensors which is of outstanding importance all throughuot the part, this would be a good time to consult Chapter 14 before tackling the present part.

Chapter 8 lays the foundations of the statics of continuous media of 3 dimensions by making our first move in the tensorial calculus and elasticity. Modeling the internal forces leads to the concept of the stress tensor based on Cauchy's tetrahedron theorem and obeying local equilibrium equations. Next, we generalize the concept of the torsor to a continuum. Usual three-dimensional (3D) bodies of which the behavior is represented by a stress torsor are called Cauchy's continua.

Chapter 9 tackles the elasticity and elementary theory of beams. To describe the kinematics of elastic bodies, we introduce the displacement vector and the strain tensor obeying Saint–Venant compatibility conditions. Next, we state Hooke's law for 3D bodies and study in particular the structure of the elasticity tensor for isotropic materials. The elastic beams are analyzed, merging displacement and stress methods and introducing the concept of transversely rigid body.

Chapter 10 is devoted to the dynamics of continuous media of 3 dimensions. After modeling their motion, we shed a new light on the equations of motion of particles and rigid bodies introduced in Chapters 3 and 5 due to the covariant derivative and the affine tensor calculus. Next, we introduce the stress-mass tensor, reveal its structure and show that it is governed by Euler's equations of motion, the cornerstone of elementary mechanics of fluids. Finally, we lay the foundations of constitutive equations with illustrations to hyperelastic materials and barotropic fluids.

Chapter 11 allows us to model all the intermediate continua between the particle trajectory of 1 dimension and the bulky body of 3 dimensions. Although general balance equations are proposed for continua of arbitrary dimensions perceived as Cosserat media, we focus our attention on the dynamics of one-dimensional (1D) material bodies (arch if solid, flow in a pipe or jet if fluid).

Chapter 12 returns to the variational methods introduced in Chapter 6, proposing an action principle for the dynamics of continua. In order to recover the balance equations, we use a special form of the calculus of variation consisting of performing variations not only on the value of the field but also on the variable.

Chapter 13 is devoted to the thermodynamics of reversible and dissipative continua. The cornerstone idea is to add to the space-time an extra dimension linked, roughly speaking, to the energy. The status of the temperature is a vector. The cornerstone tensors are its gradient called friction and the corresponding momentum tensor. For reversible processes, introducing Planck's potential reveals its structure and allows us to deduce classical potentials, internal energy, free energy and the specific entropy. The modeling of the dissipative continua is based on an additive decomposition of the momentum tensor into reversible and irreversible parts. The first principle of thermodynamics claims that it is covariant divergence free. The second principle is based on a tensorial expression of the local production of entropy. The constitutive laws are briefly discussed in the context of thermodynamics and illustrated by Navier–Stokes equations.

I.2.3. *Part 3: advanced topics*

The third part is devoted to research topics. The readers are asked whether they know the classical tools of differential geometry, some of them being recalled in Chapter 18.

In Chapter 15, the tangent space to a manifold is equipped with a differential affine structure by enhancing the concept of chart, due to a set of one parameter smooth families of charts, called a film library. In particular, we show how the fields of points of the affine tangent space can be viewed as differential operators on the scalar fields. So, we recover the concept of particle derivative, usual in the mechanics of continua.

In Chapter 16, the affine structure is enriched by Galilean, Bargmannian and Poincarean structures allowing us to derive the equation of motion in a covariant form compatible with the classical mechanics. Besides the torsors widely used in the former two parts, we introduce a new affine tensor relevant for mechanics called momentum tensor. We determine the most general transformation law of Galilean momenta. We deduce the Galilean coordinate systems from the study of the corresponding G-structure and we calculate the Galilean curvature tensor. The end of the chapter is devoted to torsor and momentum affine tensors for Bargmanian and Poincarean structures and to the underlined geometric structure of Lie group statistical mechanics.

In Chapter 17, the affine mechanics is discussed with respect to the symplectic structure on the manifold. In the framework of the coadjoint orbit method, the main concepts are the symplectic action of a group and the momentum map, allowing us to give a modern version of Noether's theorem. Bargmann's group, introduced in Chapter 13 by heuristic arguments, is now constructed as a link to the symplectic cohomology. Finally, we construct a symplectic form based on the factorization of the connection 1-form and the differential of the momentum tensor.

I.3. Historical background and key concepts

Before starting, let us give some words to briefly explain the key concepts underlying the structure of the book. The present section is addressed to experts and can be bypassed, in an initial reading, by undergraduate and postgraduate students.

General relativity is not solely a theory of gravitation which is reduced to the prediction of tiny effects such as bending of light and corrections to mercury's orbital precession but – maybe above all – it is a consistent framework for mechanics and physics of continua. It is organized around some key-ideas:

– the space-time, equipped with a metrics which makes it a Riemannian manifold;

– a symmetry group, Poincaré's one;

– associated with this group, a connection which is identified to the gravitation and of which the potentials are the 10 components of the metrics;

– a stress-energy tensor, representing the matter and divergence free;

– its identification to a tensor linked to the curvature of the manifold provides the equations allowing us to determine the 10 potentials.

More details can be found in Souriau's book "Géométrie et relativité" [SOU 08] or in the survey "Gravitation" by Misner, Thorne and Wheeler [MIS 73].

Is this scheme transposable to classical mechanics? The idea is not new and many researchers tried their hand at doing it, among them, for instance Souriau [SOU 07, SOU 97], Küntzle [KUN 72], Duval and Horváthy [DUV 85, DUV 91]. Let us draft the rough outline of this approach:

– working in the space-time but with another symmetry group, Galileo's one;

–  it preserves no metrics, then tensorial indices may be neither lowered nor raised;

– the associated connection, structured into gravity and spinning, leads to a covariant form of the equation of motion and derives from 4 potential;

– Galileo's and Poincaré's groups are both subgroups of the affine group, from which follows the idea of identifying the common elements of classical and relativistic theories: affine mechanics [SOU 97];

– it hinges on torsor, a divergence free skew-symmetric 2-contravariant affine tensor [DES 03].

The moment of a force, due to Archimedes, is a fundamental concept of mechanical science. Its modeling by means of standard mathematical tools is well known. In the modern literature, it sometimes appears under the axiomatic form of the concept of a torsor [PER 53], an object composed of a vector and a moment, endowed with the property of equiprojectivity and obeying a specific transport law. Although the latter invokes a translation of the origin, very little interest has been taken in wondering about the affine nature of this object. These elementary notions can be presented with a minimal background of vector calculus. At a higher mathematical level, another, no-lesser overlooked keystone of the mechanics is the concept of a continuous medium, especially organized around the tensorial calculus which arises from Cauchy's works about the stresses [CAU 23, CAU 27]. The general rules of this calculus were introduced by Ricci-Curbastro and Levi-Civita [RIC 01]. They are concerned by the tensors that we will call "linear tensors" insofar as their components are modified by means of linear frame changes, then of regular linear transformations, elements of the linear group. The use of moving frames