STATE ESTIMATION FOR ROBOTICS

TIMOTHY D. BARFOOT

CAMBRIDGE

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TIMOTHY D. BARFOOT
University of Toronto

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State Estimation for Robotics

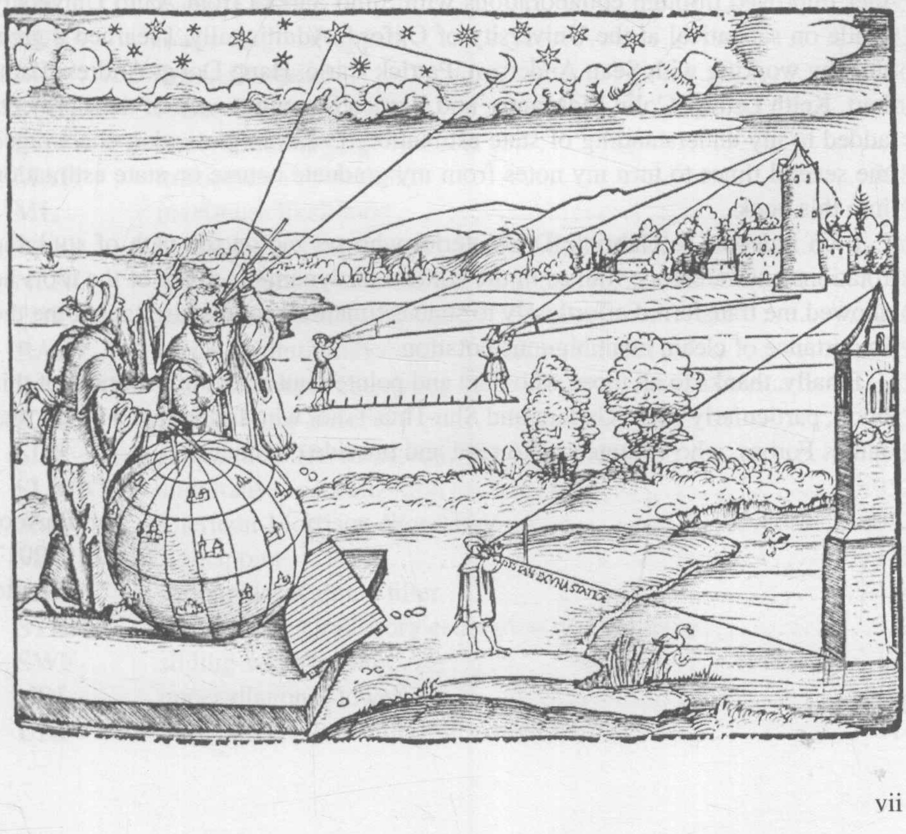
A key aspect of robotics today is estimating the state (e.g., position and orientation) of a robot as it moves through the world. Most robots and autonomous vehicles depend on noisy data from sensors such as cameras or laser rangefinders to navigate in a three-dimensional world. This book presents common sensor models and practical advice on how to carry out state estimation for rotations and other state variables. It covers both classical state estimation methods, such as the Kalman filter, and important modern topics, such as batch estimation, the Bayes filter, sigmapoint and particle filters, robust estimation for outlier rejection, and continuous-time trajectory estimation and its connection to Gaussian-process regression. The methods are demonstrated in the context of important applications, such as point-cloud alignment, pose-graph relaxation, bundle adjustment, and simultaneous localization and mapping. Students and practitioners of robotics alike will find this a valuable resource.

DR. TIMOTHY D. BARFOOT (Professor, University of Toronto Institute for Aerospace Studies – UTIAS) has been conducting research in the area of navigation of mobile robotics for more than 15 years, both in industry and academia, for applications including space exploration, mining, military, and transportation. He has made contributions in the areas of localization, mapping, planning, and control. He sits on the editorial boards of the *International Journal of Robotics Research* and the *Journal of Field Robotics* and was the General Chair of Field and Service Robotics 2015, which was held in Toronto.

Preface

My interest in state estimation stems from the field of mobile robotics, particularly for space exploration. Within mobile robotics, there has been an explosion of research referred to as *probabilistic robotics*. With computing resources becoming very inexpensive, and the advent of rich new sensing technologies, such as digital cameras and laser rangefinders, robotics has been at the forefront of developing exciting new ideas in the area of state estimation.

In particular, this field was probably the first to find practical applications of the so-called Bayes filter, a much more general technique than the famous Kalman filter. In just the last few years, mobile robotics has even started going beyond the Bayes filter to batch, nonlinear optimization-based techniques, with very promising results. Because my primary area of interest is navigation of robots in outdoor environments, I have often been faced with vehicles operating in three



Introductio Geographica by Petrus Apianus (1495–1552), a German mathematician, astronomer, and cartographer. Much of three-dimensional state estimation has to do with *triangulation* and/or *trilateration*; we measure some angles and lengths and infer the others through trigonometry.

dimensions. Accordingly, I have attempted to provide a detailed look at how to approach state estimation in three dimensions. In particular, I show how to treat rotations and poses in a simple and practical way using matrix Lie groups. The reader should have a background in undergraduate linear algebra and calculus, but otherwise, this book is fairly standalone. I hope readers of these pages will find something useful; I know I learned a great deal while creating them.

I have provided some historical notes in the margins throughout the book, mostly in the form of biographical sketches of some of the researchers after whom various concepts and techniques are named; I primarily used Wikipedia as the source for this information. Also, the first part of Chapter 6 (up to the alternate rotation parameterizations), which introduces three-dimensional geometry, is based heavily on notes originally produced by Chris Damaren at the University of Toronto Institute for Aerospace Studies.

This book would not have been possible without the collaborations of many fantastic graduate students along the way. Paul Furgale's PhD thesis extended my understanding of matrix Lie groups significantly by introducing me to their use for describing poses; this led us on an interesting journey into the details of transformation matrices and how to use them effectively in estimation problems. Paul's later work led me to become interested in continuous-time estimation. Chi Hay Tong's PhD thesis introduced me to the use of Gaussian processes in estimation theory, and he helped immensely in working out the details of the continuous-time methods presented herein; my knowledge in this area was further improved through collaborations with Simo Särkkä from Aalto University while on sabbatical at the University of Oxford. Additionally, I learned a great deal by working with Sean Anderson, Patrick Carle, Hang Dong, Andrew Lambert, Keith Leung, Colin McManus, and Braden Stenning; each of their projects added to my understanding of state estimation. Colin, in particular, encouraged me several times to turn my notes from my graduate course on state estimation into this book.

I am indebted to Gabriele D'Eleuterio, who set me on the path of studying rotations and reference frames in the context of dynamics; many of the tools he showed me transferred effortlessly to state estimation, and he also taught me the importance of clean, unambiguous notation.

Finally, thanks to all those who read and pointed out errors in the drafts of this book, particularly Marc Gallant and Shu-Hua Tsao, who found many typos, and James Forbes, who volunteered to read and provide comments.

Tim Barfoot
June 12, 2017
Toronto

Acronyms and Abbreviations

BA	bundle adjustment	327
BCH	Baker-Campbell-Hausdorff	221
BLUE	best linear unbiased estimate	68
CRLB	Cramér-Rao lower bound	113
DARCES	data-aligned rigidity-constrained exhaustive search	154
EKF	extended Kalman filter	96
GP	Gaussian process	32
GPS	Global Positioning System	3
ICP	iterative closest point	287
IEKF	iterated extended Kalman filter	101
IMU	inertial measurement unit	201
IRLS	iteratively reweighted least squares	158
ISPKF	iterated sigmapoint Kalman filter	119
KF	Kalman filter	35
LDU	lower-diagonal-upper	23
LG	linear-Gaussian	36
LTI	linear time-invariant	80
LTV	linear time-varying	74
MAP	maximum a posteriori	338
ML	maximum likelihood	339
NASA	National Aeronautics and Space Administration	96
NLNG	nonlinear, non-Gaussian	88
PDF	probability density function	256
RAE	range-azimuth-elevation	199
RANSAC	random sample consensus	155
RTS	Rauch-Tung-Striebel	53
SDE	stochastic differential equation	349
SLAM	simultaneous localization and mapping	353
SMW	Sherman-Morrison-Woodbury	23
SP	sigmapoint	105
SPKF	sigmapoint Kalman filter	114
STEAM	simultaneous trajectory estimation and mapping	353
SWF	sliding-window filter	138
UDL	upper-diagonal-lower	24
UKF	unscented Kalman filter (also called SPKF)	114

Notation

General Notation

a	Font used for quantities that are real scalars
$\mathbf{a}$	Font used for quantities that are real column vectors
$\mathbf{A}$	Font used for quantities that are real matrices
$\boldsymbol{A}$	Font used for time-invariant system quantities
$p(\mathbf{a})$	Probability density of $\mathbf{a}$
$p(\mathbf{a}\vert\mathbf{b})$	Probability density of $\mathbf{a}$ given $\mathbf{b}$
$\mathcal{N}(\mathbf{a}, \mathbf{B})$	Gaussian probability density with mean $\mathbf{a}$ and covariance $\mathbf{B}$
$\mathcal{GP}(\boldsymbol{\mu}(t), \mathcal{K}(t, t'))$	Gaussian process with mean function, $\boldsymbol{\mu}(t)$, and covariance function, $\mathcal{K}(t, t')$
$\mathcal{O}$	Observability matrix
$(\cdot)_k$	Value of a quantity at timestep k
$(\cdot)_{k_1:k_2}$	Set of values of a quantity from timestep k_1 to timestep k_2, inclusive
$\underrightarrow{\mathcal{F}}_a$	A vectrix representing a reference frame in three dimensions
$\underrightarrow{a}$	A vector quantity in three dimensions
$(\cdot)^\times$	Cross-product operator, which produces a skew-symmetric matrix from a 3×1 column
$\mathbf{1}$	Identity matrix
$\mathbf{0}$	Zero matrix
$\mathbb{R}^{M \times N}$	Vectorspace of real $M \times N$ matrices
$\hat{(\cdot)}$	A posterior (estimated) quantity
$\check{(\cdot)}$	A priori quantity

Matrix-Lie-Group Notation

$SO(3)$	The special orthogonal group, a matrix Lie group used to represent rotations
$\mathfrak{so}(3)$	The Lie algebra associated with $SO(3)$
$SE(3)$	The special Euclidean group, a matrix Lie group used to represent poses
$\mathfrak{se}(3)$	The Lie algebra associated with $SE(3)$
$(\cdot)^\wedge$	An operator associated with the Lie algebra for rotations and poses

$(\cdot)^{\curlywedge}$	An operator associated with the adjoint of an element from the Lie algebra for poses
$\mathrm{Ad}(\cdot)$	An operator producing the adjoint of an element from the Lie group for rotations and poses
$\mathrm{ad}(\cdot)$	An operator producing the adjoint of an element from the Lie algebra for rotations and poses
$\mathbf{C}_{ba}$	A 3×3 rotation matrix (member of $SO(3)$) that takes points expressed in $\underrightarrow{\mathcal{F}}_a$ and reexpresses them in $\underrightarrow{\mathcal{F}}_b$, which is rotated with respect to $\underrightarrow{\mathcal{F}}_a$
$\mathbf{T}_{ba}$	A 4×4 transformation matrix (member of $SE(3)$) that takes points expressed in $\underrightarrow{\mathcal{F}}_a$ and reexpresses them in $\underrightarrow{\mathcal{F}}_b$, which is rotated/translated with respect to $\underrightarrow{\mathcal{F}}_a$
$\boldsymbol{\mathcal{T}}_{ba}$	A 6×6 adjoint of a transformation matrix (member of $\mathrm{Ad}(SE(3))$)

Contents

1

Introduction

Robotics inherently deals with things that move in the world. We live in an era of rovers on Mars, drones surveying the Earth, and, soon, self-driving cars. And although specific robots have their subtleties, there are also some common issues we must face in all applications, particularly *state estimation* and *control*.

The *state* of a robot is a set of quantities, such as position, orientation, and velocity, that, if known, fully describe that robot's motion over time. Here we focus entirely on the problem of estimating the state of a robot, putting aside the notion of control. Yes, control is essential, as we would like to make our robots behave in a certain way. But the first step in doing so is often the process of determining the state. Moreover, the difficulty of state estimation is often underestimated for real-world problems, and thus it is important to put it on an equal footing with control.

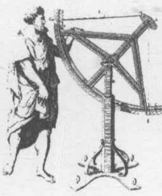
In this book, we introduce the classic estimation results for linear systems corrupted by Gaussian measurement noise. We then examine some of the extensions to nonlinear systems with non-Gaussian noise. In a departure from typical estimation texts, we take a detailed look at how to tailor general estimation results to robots operating in three-dimensional space, advocating a particular approach to handling rotations.

The rest of this introduction provides a little history of estimation, discusses types of sensors and measurements, and introduces the problem of state estimation. It concludes with a breakdown of the contents of the book and provides some other suggested reading.

1.1 A Little History

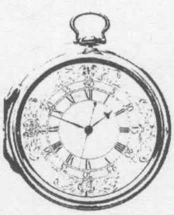
About 4,000 years ago, the early seafarers were faced with a vehicular state estimation problem: how to determine a ship's position while at sea. Early attempts to develop primitive charts and make observations of the sun allowed local navigation along coastlines. However, it was not until the fifteenth century that global navigation on the open sea became possible with the advent of key technologies and tools. The mariner's compass, an early form of the magnetic compass, allowed crude measurements of direction to be made. Together with coarse nautical charts, the compass made it possible to sail along rhumb lines between key destinations (i.e., following a compass bearing). A series of instruments was then gradually invented that made it possible to measure the angle between distant points (i.e., cross-staff, astrolabe, quadrant, sextant, theodolite) with increasing accuracy (Figure 1.1).

Figure 1.1 Quadrant. A tool used to measure angles.



These instruments allowed latitude to be determined at sea fairly readily using celestial navigation. For example, in the Northern Hemisphere, the angle between the North Star, Polaris, and the horizon provides the latitude. Longitude, however, was a much more difficult problem. It was known early on that an accurate timepiece was the missing piece of the puzzle for the determination of longitude. The behaviours of key celestial bodies appear differently at different locations on the Earth. Knowing the time of day therefore allows longitude to be inferred. In 1764, British clockmaker John Harrison built the first accurate portable timepiece that effectively solved the longitude problem; a ship's longitude could be determined to within about 10 nautical miles (Figure 1.2).

Figure 1.2 Harrison's H4. The first clock able to keep accurate time at sea, enabling determination of longitude.



Estimation theory also finds its roots in astronomy. The method of least squares was pioneered[1] by Gauss, who developed the technique to minimize the impact of measurement error in the prediction of orbits. Gauss reportedly used least squares to predict the position of the dwarf planet Ceres after passing behind the Sun, accurate to within half a degree (about nine months after it was last seen). The year was 1801, and Gauss was 23. Later, in 1809, he proved that the least squares method is optimal under the assumption of normally distributed errors. Most of the classic estimation techniques in use today can be directly related to Gauss' least squares method.

The idea of fitting models to minimize the impact of measurement error carried forward, but it was not until the middle of the twentieth century that estimation really took off. This was likely correlated with the dawn of the computer age. In 1960, Kalman published two landmark papers that have defined much of what has followed in the field of state estimation. First, he introduced the notion of *observability* (Kalman, 1960a), which tells us when a state can be inferred from a set of measurements in a dynamic system. Second, he introduced an optimal framework for estimating a system's state in the presence of measurement noise (Kalman, 1960b); this classic technique for linear systems (whose measurements are corrupted by Gaussian noise) is famously known as the *Kalman filter* and has been the workhorse of estimation for the more than 50 years since its inception. Although used in many fields, it has been widely adopted in aerospace applications. Researchers at the National Aeronautics and Space Administration (NASA) were the first to employ the Kalman filter to aid in the estimation of spacecraft trajectories on the Ranger, Mariner, and Apollo programs. In particular, the on-board computer on the Apollo 11 lunar module, the first manned spacecraft to land on the surface of the Moon, employed a Kalman filter to estimate the module's position above the lunar surface based on noisy radar measurements.

Carl Friedrich Gauss (1777–1855) was a German mathematician who contributed significantly to many fields, including statistics and estimation.

Rudolf Emil Kalman (1930–2016) was a Hungarian-born American electrical engineer, mathematician, and inventor.

[1] There is some debate as to whether Adrien Marie Legendre might have come up with least squares before Gauss.

Many incremental improvements have been made to the field of state estimation since these early milestones. Faster and cheaper computers have allowed much more computationally complex techniques to be implemented in practical systems. However, until about 15 years ago, it seemed that estimation was possibly waning as an active research area. But something has happened to change that; exciting new sensing technologies are coming along (e.g., digital cameras, laser imaging, the *Global Positioning System* (GPS) satellites) that pose new challenges to this old field.

EARLY ESTIMATION MILESTONES

1654	Pascal and Fermat lay foundations of probability theory
1764	Bayes' rule
1801	Gauss uses least squares to estimate the orbit of the planetoid Ceres
1805	Legendre publishes "least squares"
1913	Markov chains
1933	(Chapman)-Kolmogorov equations
1949	Wiener filter
1960	Kalman (Bucy) filter
1965	Rauch-Tung-Striebel smoother
1970	Jazwinski coins "Bayes filter"

1.2 Sensors, Measurements, and Problem Definition

To understand the need for state estimation is to understand the nature of sensors. All sensors have a limited precision. Therefore, all measurements derived from real sensors have associated uncertainty. Some sensors are better at measuring specific quantities than others, but even the best sensors still have a degree of imprecision. When we combine various sensor measurements into a state estimate, it is important to keep track of all the uncertainties involved and therefore (it is hoped) know how confident we can be in our estimate.

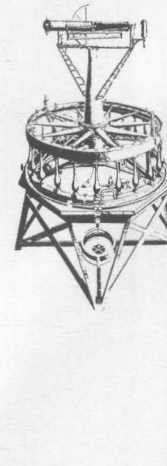
In a way, state estimation is about doing the best we can with the sensors we have. This, however, does not prevent us from, in parallel, improving the quality of our sensors. A good example is the *theodolite* sensor that was developed in 1787 to allow triangulation across the English Channel (Figure 1.3). It was much more precise than its predecessors and helped show that much of England was poorly mapped by tying measurements to well-mapped France.

It is useful to put sensors into two categories: *interoceptive*[2] and *exteroceptive*. These are actually terms borrowed from human physiology, but they have become somewhat common in engineering. Some definitions follow:[3]

in·tero·cep·tive [int-ə-rō-'sep-tiv], *adjective*: of, relating to, or being stimuli arising within the body.

ex·tero·cep·tive [ek-stə-rō-'sep-tiv], *adjective*: relating to, being, or activated by stimuli received by an organism from outside.

Figure 1.3 Theodolite. A better tool to measure angles.



Typical interoceptive sensors are the accelerometer (measures translational acceleration), gyroscope (measures angular rate), and wheel odometer (measures angular rate). Typical exteroceptive sensors are the camera (measures range/bearing to a landmark or landmarks) and time-of-flight transmitter/receiver (e.g., laser rangefinder, pseudolites, GPS transmitter/receiver). Roughly speaking, we can think of exteroceptive measurements as being of the position and orientation of a vehicle, whereas interoceptive ones are of a vehicle's velocity or acceleration. In most cases, the best state estimation concepts make use of both interoceptive and exteroceptive measurements. For example, the combination of

2 Sometimes *proprioceptive* is used synonomously.

3 Merriam-Webster's Dictionary.

a GPS receiver (exteroceptive) and an inertial measurement unit (three linear accelerometers and three rate gyros; interoceptive) is a popular means of estimating a vehicle's position/velocity on Earth. And the combination of a Sun/star sensor (exteroceptive) and three rate gyros (interoceptive) is commonly used to carry out pose determination on satellites.

Now that we understand a little bit about sensors, we are prepared to define the problem investigated in this book:

Estimation *is the problem of reconstructing the underlying state of a system given a sequence of measurements as well as a prior model of the system.*

There are many specific versions of this problem and just as many solutions. The goal is to understand which methods work well in which situations, in order to pick the best tool for the job.

1.3 How This Book Is Organized

The book is broken into three main parts:

I Estimation Machinery
II Three-Dimensional Machinery
III Applications

The first part, "Estimation Machinery," presents classic and state-of-the-art estimation tools, without the complication of dealing with things that live in three-dimensional space (and therefore translate and rotate); the state to be estimated is assumed to be a generic vector. For those not interested in the details of working in three-dimensional space, this first part can be read in a standalone manner. It covers both recursive state estimation techniques and batch methods (less common in classic estimation books). As is commonplace in robotics and machine learning today, we adopt a *Bayesian* approach to estimation in this book. We contrast (full) Bayesian methods with *maximum a posteriori* (MAP) methods and attempt to make clear the difference between these when faced with nonlinear problems. The book also connects continuous-time estimation with Gaussian process regression from the machine-learning world. Finally, it touches on some practical issues, such as robust estimation and biases.

The second part, "Three-Dimensional Machinery," provides a basic primer on three-dimensional geometry and gives a detailed but accessible introduction to matrix Lie groups. To represent an object in three-dimensional space, we need to talk about that object's translation and rotation. The rotational part turns out to be a problem for our estimation tools because rotations are not *vectors* in the usual sense and so we cannot naively apply the methods from Part I to three-dimensional robotics problems involving rotations. Part II, therefore, examines the geometry, kinematics, and probability/statistics of rotations and poses (translation plus rotation).

Finally, in the third part, "Applications," the first two parts of the book are brought together. We look at a number of classic three-dimensional estimation problems involving objects translating and rotating in three-dimensional space. We show how to adapt the methods from Part I based on the knowledge gained in Part II. The result is a suite of easy-to-implement methods for three-dimensional state estimation. The spirit of these examples can also, we hope, be adapted to create other novel techniques moving forward.

1.4 Relationship to Other Books

There are many other good books on state estimation and robotics, but very few cover both topics simultaneously. We briefly describe a few recent works that do cover these topics and their relationships to this book.

Probabilistic Robotics by Thrun et al. (2006) is a great introduction to mobile robotics, with a large focus on state estimation in relation to mapping and localization. It covers the probabilistic paradigm that is dominant in much of robotics today. It mainly describes robots operating in the two-dimensional, horizontal plane. The probabilistic methods described are not necessarily limited to the two-dimensional case, but the details of extending to three dimensions are not provided.

Computational Principles of Mobile Robotics by Dudek and Jenkin (2010) is a great overview book on mobile robotics that touches on state estimation, again in relation to localization and mapping methods. It does not work out the details of performing state estimation in three dimensions.

Mobile Robotics: Mathematics, Models, and Methods by Kelly (2013) is another excellent book on mobile robotics and covers state estimation extensively. Three-dimensional situations are covered, particularly in relation to satellite-based and inertial navigation. As the book covers all aspects of robotics, it does not delve deeply into how to handle rotational variables within three-dimensional state estimation.

Robotics, Vision, and Control by Corke (2011) is another great and comprehensive book that covers state estimation for robotics, including in three dimensions. Similarly to the previously mentioned book, the breadth of Corke's book necessitates that it not delve too deeply into the specific aspects of state estimation treated herein.

Bayesian Filtering and Smoothing by Särkkä (2013) is a super book focused on recursive Bayesian methods. It covers the recursive methods in far more depth than this book but does not cover batch methods nor focus on the details of carrying out estimation in three dimensions.

Stochastic Models, Information Theory, and Lie Groups: Classical Results and Geometric Methods by Chirikjian (2009), an excellent two-volume work, is perhaps the closest in content to the current book. It explicitly investigates the consequences of carrying out state estimation on matrix Lie groups (and hence rotational variables). It is quite theoretical in nature and goes beyond the current book in this sense, covering applications beyond robotics.

Engineering Applications of Noncommutative Harmonic Analysis: With Emphasis on Rotation and Motion Groups by Chirikjian and Kyatkin (2001) and the recent update, *Harmonic Analysis for Engineers and Applied Scientists: Updated and Expanded Edition* (Chirikjian and Kyatkin, 2016), also provide key insights to representing probability globally on Lie groups. In the current book, we limit ourselves to approximate methods that are appropriate to the situation where rotational uncertainty is not too high.

Although it is not an estimation book per se, it is worth mentioning *Optimization on Matrix Manifolds* by Absil et al. (2009), which provides a detailed look at how to handle optimization problems when the quantity being optimized is not necessarily a vector, a concept that is quite relevant to robotics because rotations do not behave like vectors (they form a Lie group).

The current book is somewhat unique in focusing only on state estimation and working out the details of common three-dimensional robotics problems in enough detail to be easily implemented for many practical situations.