PIECEWISE LINEAR STRUCTURES ON TOPOLOGICAL MANIFOLDS

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Dedicated to Irina and Marina

Preface

The study of triangulations of topological spaces has always been at the root of geometric topology. Among the most studied triangulations are piecewise linear triangulations of high-dimensional topological manifolds. Their study culminated in the late 1960s–early 1970s in a complete classification in the work of Kirby and Siebenmann. It is this classification that we discuss in this book, including the celebrated Hauptvermutung and Triangulation Conjecture.

The goal of this book is to provide a readable and well-organized exposition of the subject, which would be suitable for advanced graduate students in topology. An exposition like this is currently lacking. The foundational monograph of Kirby and Siebenmann [KS2] proving the classification was written on the heels of the proof. It contains all the necessary ingredients but, written in the form of essays, can hardly serve as an exposition. Another very useful source of information on the subject, the book of Ranicki [Ran], has the same drawback as being a collection of research papers.

In this book, I attempted to give a panoramic view of the theory. Given in how many different directions this theory branches out, I took special care not to lose sight of the forest for the trees. For instance, I chose to merely state several well-known theorems, providing references to well-written proofs available in the literature.

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Yuli Rudyak, *Gainesville, Florida*, November 2015

Introduction

Throughout this volume we use abbreviation PL for "piecewise linear".

For introduction to PL topology, including definitions of simplicial complexes, polyhedra, PL maps, etc. see [Hud, RS]

Hauptvermutung (main conjecture) is an abbreviation for *die Hauptvermutung der kombinatorischen Topologie* (the main conjecture of combinatorial topology). It seems that the conjecture was formulated in the papers of Steinitz [Ste] and Tietze [Ti] in 1908. This is also stated in [AH].

The conjecture states that the topology of a simplicial complex determines completely its combinatorial structure. In other words, two simplicial complexes are simplicially isomorphic whenever they are homeomorphic. This conjecture was disproved by Milnor [Mi2] in 1961. In fact, Milnor found a pair of homeomorphic simplicial complexes such that the Whitehead torsion of this pair is non-trivial. See Cohen [C] for a textbook account.

Note, however, that the Whitehead torsion cannot distinguish homeomorphic *manifolds*, [KS1, C]. Thus, in case of manifolds, one can propose a refined version of the *Hauptvermutung* by considering simplicial complexes with natural additional restrictions. A *combinatorial triangulation* is defined to be a simplicial complex such that the star of every point (the union of all closed simplexes containing the point) is simplicially isomorphic to the n-dimensional ball. A *PL manifold*, or *combinatorial manifold* is defined to be a topological manifold M together with a homeomorphism $M \to K$ where K is a combinatorial triangulation. Equivalently, a PL manifold can also be defined as a manifold equipped with a maximal PL atlas.

There exist topological manifolds that are homeomorphic to a simplicial complex but do not admit a PL structure (non-combinatorial triangulations), see Example 3.5.5. Furthermore, there exist topological manifolds that are not homeomorphic to any simplicial complex, see Example 3.5.7.

Now, the *Hauptvermutung for manifolds* is the conjecture that any two homeomorphic PL manifolds are PL homeomorphic. The related *Combinatorial Triangulation Conjecture* states that every topological manifold admits a PL structure, i.e., can be triangulated by a PL manifold. Both these conjectures were disproved by Kirby and Siebenmann [Sieb4, KS1, KS2]. In fact, Kirby and Siebenmann classified PL structures on high-dimensional ($\geqslant 5$) topological manifolds. It turned out that a topological manifold can have different (not PL homeomorphic, non-concordant) PL structures, as well as having no PL structures. Now we give a brief description of these results.

In dimensionals $\leqslant 3$ every topological manifold admits a PL structure that is unique up to PL homeomorphism, see [Rad, P, Mo]. The classifcation of PL structures on 4-dimenaional topological manifolds is not completed yet, cf. [FQ, K2].

Let $BTOP$ and BPL be the classifying spaces for stable topological and PL (micro)bundles, respectively. We regard the forgetful map

$$\alpha : BPL \to BTOP$$

as a fibration and denote its homotopy fiber by TOP/PL.

Let $f : M \to BTOP$ classify the stable tangent bundle of a topological manifold M. By the main properties of classifying spaces, every PL structure on M gives us an α-lifting of f and that every two such liftings for the same PL structure are fiberwise homotopic.

It is remarkable that the converse is also true if $\dim M \geqslant 5$, see [LR1, KS2]. In greater detail, M admits a PL structure if f admits an α-lifting (*the Existence Theorem* 1.7.4), and concordance classes of PL structures on M are in a bijective correspondence with fiberwise homotopy classes of α-liftings of f (*the Classification Theorem* 1.7.2). So, the homotopy information on the space TOP/PL is extremely useful in PL classifying of topological manifolds. Fortunately, Kirby and Siebenmann have made great progress there: they proved the following

Main Theorem: *There is a homotopy equivalence*

$$TOP/PL \simeq K(\mathbb{Z}/2, 3).$$

Thus, there is at most one possible obstruction

$$\varkappa(M) \in H^4(M;\pi_3(TOP/PL)) = H^4(M;\mathbb{Z}/2)$$

to an α-lifting of the map f.

In particular, a topological manifold $M, \dim M \geqslant 5$ admits a PL structure if and only if $\varkappa(M) = 0$. Furthermore, the set of fiberwise homotopic α-liftings of f (if they exist) is in a bijective correspondence with $H^3(M;\mathbb{Z}/2)$. At manifolds level, we can say that every homeomorphism $h : V \to M$ of a PL manifold V yields a class

$$\varkappa(h) \in H^3(M;\mathbb{Z}/2),$$

and $\varkappa(h) = 0$ if and only if h is concordant to the identity map 1_M. Moreover, every class $a \in H^3(M;\mathbb{Z}/2)$ has the form $a = \varkappa(h)$ for some homeomorphism $h : V \to M$ of two PL manifolds.

These results yield the complete classification of PL structures on a topological manifold of dimension $\geqslant 5$. In particular, the situation with *Hauptvermutung* turns out to be understandable. See Section 3.4 for more detailed exposition.

We would like to explain the following. It can happen that non-concordant PL structures on M yield PL homeomorphic PL manifolds (like that two p-liftings $f_1, f_2 : M \to BPL$ of f can be non-fiberwise homotopic). Indeed, a PL map $M \to M$ of a PL manifold M can turn the atlas into a non-concordant to the original one, see Example 3.5.3. So, in fact, the set of pairwise non-concordant PL manifolds which are homeomorphic to a given PL manifold is in a bijective correspondence with the set $H^3(M;\mathbb{Z}/2)/R$ where R is the following equivalence relation: two concordance classes of PL structures are equivalent if the corresponding PL manifolds are PL homeomorphic. The *Hauptvermutung* for manifolds states that the set $H^3(M;\mathbb{Z}/2)/R$ is a singleton for all M. But this is wrong in general.

Namely, there exists a PL manifold M which is homeomorphic but not PL isomorphic to $\mathbb{RP}^n, n \geqslant 5$, see Example 3.5.1. So, here we have a counterexample to the *Hauptvermutung*.

To complete the picture, we mention again that there are topological manifolds that do not admit any PL structure, see Example 3.5.4. Moreover, there are manifold that cannot be triangulated as simplicial complexes, see Example 3.5.7.

Recall that every smooth manifold admits a canonical PL structure [Cai, W], while every PL manifold is, tautologically, a topological manifolds. Now we compare the classes of smooth, PL and topological manifolds, and see that there is a big difference between first and second classes, and not so big difference between second and third ones. From the homotopy-theoretical point of view, one can say that the space PL/O (which classifies smooth structures on PL manifold, see Remark 1.7.8) has many non-trivial homotopy groups, while the space TOP/PL is an Eilenberg–MacLane space. Geometrically, one can mention that there are many smooth manifolds which are PL homeomorphic to standard sphere S^n but pairwise non-diffeomorphic [KM], while any PL manifold $M^n, n \geqslant 5$ is PL homeomorphic to S^n provided that M is homeomorphic to S^n, [Sma].

It is interesting and worthwhile to go one step deeper and explain the following. Recall that a manifold M is called *almost parallelizable* if M becomes parallelizable after deletion of a point. Let σ_k^S (resp. σ_k^{PL}, resp. σ_k^{TOP}) denote the minimal positive integer number which can be realized as the signature of the closed smooth (resp. PL, resp. topological) almost parallelizable $4k$-dimensional manifold. Clearly, $\sigma_k^S \geqslant \sigma_k^{PL} \geqslant \sigma_k^{TOP}$.

Let B_m denote the mth Bernoulli numbers, see [Wash] (we use the even index notation, i.e., $B_{2n+1} = 0$). It turns out to be that

$$\sigma_1^S = 16 \text{ and } \sigma_k^S = 2^{2k+1}(2^{2k-1} - 1) \text{ numerator } (4B_{2k}/k) \text{ for } k > 1.$$

See [Ro] for $k = 1$ and [MK] or [MS, Appendix B] for $k > 1$. In particular, σ_k^S strictly increases with respect to k.

Concerning the numbers σ^{PL} and σ^{TOP}, it turns out to be that

$$\sigma_1^{PL} = 16 \text{ and } \sigma_k^{PL} = 8 \text{ for all } k > 1,$$

and

$$\sigma_k^{TOP} = 8 \text{ for all } k.$$

First, for all k the number 8 divides the number σ_k by purely algebraic reasons, [Br2, Proposition III.1.4]. Furthermore, $\sigma_1^{PL} = \sigma_1^S = 16$ since there is no difference between PL and smooth cases up to dimension 6, see 1.7.8. Let W^{4k} be a $4k$-dimensional smooth manifold with boundary (Milnor's pumbing) described in [Br2, Theorem V.2.1]. This is a parallelizable manifold of signature 8. Furthermore, for $k > 1$ the boundary ∂W^{4k} is a homotopy sphere. Hence, ∂W^{4k} is PL homeomorphic to the standard sphere by the Smale Theorem [Sma]. So, the cone $C = C(\partial W^{4k})$ is PL homeomorphic to the standard disk, and we get a closed almost parallelizable PL manifold $W^{4k} \cup_{\partial W^{4k}} C$ of signature 8.

To prove that $\sigma_1^{TOP} = 8$, consider the plumbing $W = W^4$ as above. Now its boundary ∂W is not simply-connected, but it is a homology 3-sphere. Freedman

[F, Theorem 1.4′] proved that ∂W bounds a contractible topological 4-manifold P (in fact, this holds for any homology 3-sphere). Now, the the space

$$W \cup_{\partial W} P$$

is a closed almost parallelizable topology manifold of signature 8.

So, $\sigma_k^{TOP} = \sigma_k^{PL} > \sigma_k^S$ for $k > 1$, and we see again that there is a big difference between smooth and PL cases and not so big difference between PL and topological cases. Nevertheless, the last difference does not vanish, and the numerical inequality

$$16 = \sigma_1^{PL} \neq \sigma_1^{TOP} = 8$$

occurs whenever we meet a contrast between PL and topological world. For example, we will see below that the number

$$2 = 16/8 = \sigma_1^{PL}/\sigma_1^{TOP}$$

is another guise of the number

$$2 = \text{ the order of the group } \pi_3(TOP/PL).$$

In this context, it makes sense to notice about low-dimensional manifolds, because of the following remarkable contrast. There is no difference between PL and smooth manifolds in dimension < 7: every PL manifold $V^n, n < 7$ admits a smooth structure that is unique up to diffeomorphism. However, there are infinitely many smooth manifolds which are homeomorphic to $\mathbb{R}^4$ but pairwise non-diffeomorphic, see Section 3.5, Summary.

Concerning the description of the homotopy type of TOP/PL, we have the following. Because of the Classification Theorem, if $k + n \geqslant 5$ then the group $\pi_n(TOP/PL)$ is in a bijective correspondence with the set of concordance classes of PL structures on $\mathbb{R}^k \times S^n$. However, this set (of concordance classes) looks wild and uncontrollable. In order to make the situation more manageable, we consider PL structures on the *compact* manifold $T^n \times S^k$ and then extract the necessary information about the universal covering $\mathbb{R}^n \times S^k$ from here. We can't do it directly, but there is a trick (the Reduction Theorem 1.9.7 that is based by ideas of Kirby) which allows us to estimate PL structures on $\mathbb{R}^n \times S^k$ in terms of the so-called *homotopy PL structures* on $T^n \times S^k$ (more precisely, we should consider the homotopy PL structures on $T^n \times D^k$ modulo the boundary), see Section 1.4 for the definitions. Now, using results of Hsiang and Shaneson [HS] or Wall [W3, W4] about homotopy PL structures on $T^n \times D^k$, one can prove that $\pi_i(TOP/PL) = 0$ for $i \neq 3$ and that $\pi_3(TOP/PL)$ has at most 2 elements. Finally, there exists a high-dimensional topological manifold which does not admit any PL structure (Corollary 1.8.4, Remark 1.8.6). Hence, by the Existence Theorem, the space TOP/PL is not contractible. Thus, $TOP/PL \simeq K(\mathbb{Z}/2, 3)$.

For better arrangement of the previous matter, look at the graph located after the Introduction. We formulate without proofs the boxed statements (and provide the necessary preliminaries and references) there, while in Chapter 1 we explain how a statement (box) can be deduced from others, accordingly with the arrows in the graph.

Let me comment the top box of the graph. Sullivan [Sul1, Sul2] proved that the *Hauptvermutung* holds for simply-connected closed manifolds $M, \dim M \geqslant 5$ with $H_3(M)$ 2-torsion free.

In greater detail, let G_n be the monoid of homotopy self-equivalences $S^{n-1} \to S^{n-1}$, let BG_n be the classifying space for G_n, and let $BG = \lim_{n\to\infty} BG_n$. There is an obvious forgetful map $BPL \to BG$ (delete zero section), and we denote the homotopy fiber of this map by G/PL. For every homotopy equivalence of closed PL manifolds $h : V \to M$, Sullivan defined the *normal invariant* of h to be a certain homotopy class $j_G(h) \in [M, G/PL]$, see Section 1.5.

Let $M, \dim \geqslant 5$ be a closed PL manifold such that $H_3(M)$ is 2-torsion free. Sullivan proved that, for every *homeomorphism* $h : V \to M$, we have $j_G(h) = 0$. Moreover, this theorem implies that if, in addition, M is simply-connected then h is homotopic to a PL homeomorphism. So, as we already noted, the *Hauptvermutung* holds for simply-connected closed manifolds $M, \dim M \geqslant 5$ with $H_3(M)$ 2-torsion free.

Definitely, the above-mentioned Sullivan Theorem on the Normal Invariant of a Homeomorphism is important by itself. However, here this theorem plays also an additional substantial role. Namely, the Sullivan Theorem for $T^n \times S^k$ is a lemma in classifying of homotopy structures on $T^n \times D^k$, cf. Section 1.6. For this reason we first prove the Sullivan Theorem for $T^n \times S^k$ (the top box), then use it in the proof of the Main Theorem, and then (in Chapter 2) use the Main Theorem in order to prove the Sullivan Theorem in full generality.

I decided to present a proof of the Sullivan Theorem in the volume, Section 3.3 because the exposition in [Sul2] is quite intricate.

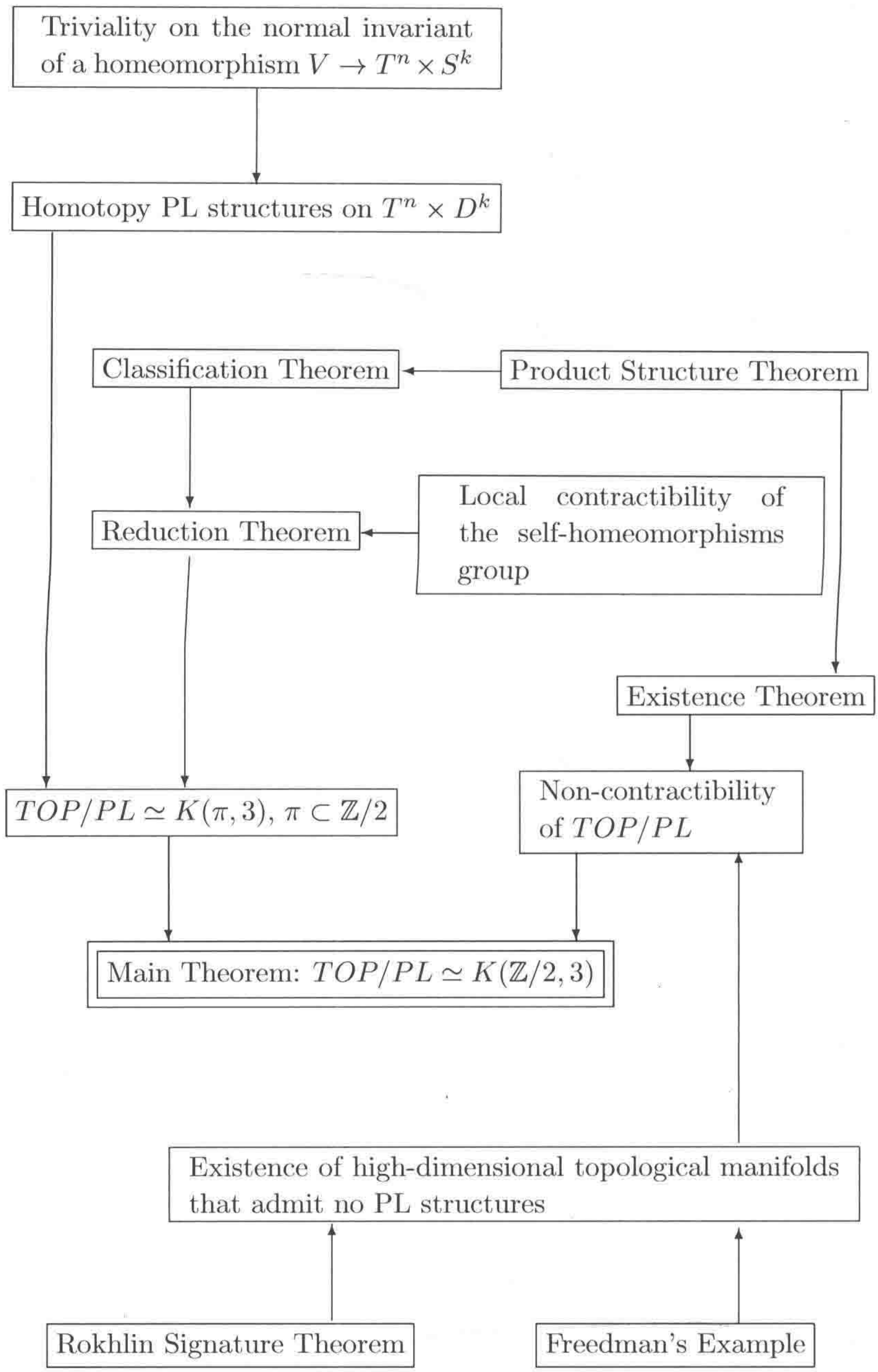
This volume is organized as follows. After the Introduction we present the above-mentioned graph, and the extensive comments of the graph appear in the first chapter. In other words, Chapter 1 contains the architecture of the proof of the Main Theorem.

The second chapter contains a proof of the Sullivan Theorem on the triviality of normal invariant of a homeomorphism for $T^n \times S^k$, i.e., we attend the top box of the graph.

The third chapter contains some applications of the Main Theorem. We complete the proof of the Sullivan Theorem on the triviality of the normal invariant of a homeomorphism in full generality. Then we tell more on classification of PL manifolds and, in particular, on Hauptvermutung. Several interesting examples are considered. Finally, we discuss the homotopy and topological invariance of certain characteristic classes.

Graph

Here I present the graph, and after the picture I list the boxed claims, with extraction the correspondent tags inside the body of the manuscript. Some minor comments are given. Some theorems here are stated simpler than those in the main text.


Triviality on the normal invariant of a homeomorphism $V \to T^n \times S^k$
Homotopy PL structures on $T^n \times D^k$
Classification Theorem
Product Structure Theorem
Reduction Theorem
Local contractibility of the self-homeomorphisms group
Existence Theorem
$TOP/PL \simeq K(\pi, 3)$, $\pi \subset \mathbb{Z}/2$
Non-contractibility of TOP/PL
Main Theorem: $TOP/PL \simeq K(\mathbb{Z}/2, 3)$
Existence of high-dimensional topological manifolds that admit no PL structures
Rokhlin Signature Theorem
Freedman's Example

1. Triviality of the normal invariant of a homeomorphism

$$V \to T^n \times S^k.$$

This is Theorem 2.8.1: If an element $x \in \mathcal{S}_{PL}(T^n \times S^k)$ can be represented by a homeomorphism $h : V \to T^k \times S^n$, then $j_G(x) = 0$.

Here and below we denote by $\mathcal{S}_{PL}(X)$ the set of equivalence classes of homotopy triangulations of a topological manifold X.

2. Homotopy PL structures on $T^n \times D^k$. This is Theorem 1.6.3: Supposed that $k + n \geqslant 5$. Then the following holds:

(i) if $k > 3$ then the set $\mathcal{S}_{PL}(T^n \times D^k)$ consists of precisely one (trivial) element;

(ii) if $k < 3$ then every element of $\mathcal{S}_{PL}(T^n \times D^k)$ can be finitely covered by the trivial element;

(iii) the set $\mathcal{S}_{PL}(T^n \times D^3)$ contains at most one element which cannot be finitely covered by the trivial element.

3. Classification Theorem. This is Theorem 1.7.2: If $\dim M \geqslant 5$ and M admits a PL structure, then the map

$$j_{TOP} : \mathcal{T}_{PL}(M) \to [M, TOP/PL]$$

is a bijection.

Here and below we denote by $\mathcal{T}_{PL}(X)$ the set of concordance classes of PL structures on a topological manifold X. The map j_{TOP} defined in 1.5.1.

4. Product Structure Theorem. This is Theorem 1.7.1: For every $n \geqslant 5$ and every $k \geqslant 0$, the map

$$e : \mathcal{T}_{PL}(M) \to \mathcal{T}_{PL}(M \times \mathbb{R}^k)$$

is a bijection. Here the map e turns a PL strucrture on M into a PL structure on $M \times \mathbb{R}^k$ in an obvious way: the product with $\mathbb{R}^k$. Roughly speaking, this theorem establishes a bijection between (the concordance classes of) PL structures on M and $M \times \mathbb{R}^k$. The Classification Theorem 1.7.2 and the Existence Theorem 1.7.4 are consequences of the Product Structure Theorem.

5. Reduction Theorem. This is Theorem 1.9.7. It reduces an evaluation of groups $\pi_i(TOP/PL)$ to the evaluation of sets $\mathcal{S}_{PL}(T^n \times D^k)$.

6. Local contractibility of the homeomorphism group. This is Theorem 1.9.1: The space of self-homeomorphisms of a compact manifold M is locally contractible.

7. $TOP/PL \simeq K(\pi, 3)$, $\pi \subset \mathbb{Z}/2$. This is Theorem 1.9.8.

8. Existence Theorem. This is Theorem 1.7.4: A topological manifold M with $\dim M \geqslant 5$ admits a PL structure if and only if the tangent bundle of M admits a PL structure.

9. Main Theorem: $TOP/PL \simeq K(\mathbb{Z}/2, 3)$. This is Theorem 1.9.9.

10. Non-contractibility of TOP/PL. This is Corollary 1.8.5.

11. Existence of high-dimensional topological manifolds that admit no PL structures. See Corollary 1.8.4 and Remark 1.8.6 for such examples.

12. Rokhlin Signature Theorem. This is Theorem 1.8.1: Let M be a closed 4-dimensional PL manifold with $w_1(M) = 0 = w_2(M)$. Then the signature of M is divisible by 16.

13. Freedman's Example. This is Theorem 1.8.2: There exists a closed simply-connected topological 4-dimensional manifold V with $w_2(V) = 0$ and the signature equal to 8. This example provides the equality $\sigma_1^{TOP} = 8$.

Actually, the original Kirby–Siebenmann proof of the Main Theorem appeared before the Freedman's example and therefore did not use the last one, see Remark 1.8.6. However, as we have seen, the inequality $\sigma_1^{PL} \neq \sigma_1^{TOP}$ clarify relations between PL and topological manifolds, and thus Freedman's example should be (and is) incorporated in the exposition of the global picture.