Wilhelm Rust

Non-Linear Finite Element Analysis in Structural Mechanics

Springer

Wilhelm Rust

Non-Linear Finite Element Analysis in Structural Mechanics

Springer

Wilhelm Rust
Faculty II, Dep. of Mechanical Engineering
University of Applied Sciences and Arts
Hannover
Germany

ISBN 978-3-319-13379-9 ISBN 978-3-319-13380-5 (eBook)
DOI 10.1007/978-3-319-13380-5
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2015933153

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Non-Linear Finite Element Analysis in Structural Mechanics

Preface

This textbook introduces into the theory of non-linear Finite Element Methods (FEM) in structural mechanics, divided into the main parts on geometric non-linearity, non-linear material behaviour and contact. While it is not possible to describe the total FEM of linear mechanics in one book, this is even more the case for the non-linear FEM, as "non-linear" is not a special property but means that the limiting assumptions, which for good reason dominate undergraduate studies in Technical Mechanics, are missing. This book should prepare the reader to work with advanced books and papers.

The formulae used are intentionally derived in detail in order to enable the reader to transfer the described relations into computer programs and to create equations for similar physical effects.

The book addresses first and foremost students who want to attain Master's level, but FEM users should get useful insights as well. In the linear FEM, provided the systems are sufficiently constrained, a result is always obtained (the correctness/accuracy is not to be discussed here); however, the user, especially the novice one, of non-linear analysis will end up in non-convergence and thus without equilibrium in a number of attempts. In this situation, it is good to know the potential causes. This will help to decide whether and how convergence can be achieved by changes to the settings. Here, the chapters on stability and on convergence in contact analysis are recommended. It should be noted that the success of a non-linear analysis depends on realistic input data, as a failure of the system will not only appear in the final results (when comparing them with strengths) but will influence convergence at an earlier stage.

For the user there is a further necessity—maybe even more important—of the theoretical background: the FEM programs on the market offer numerous options and settings to choose which usually are described for a user with knowledge on how Finite Elements are formulated. In this book, it is assumed that the reader knows how this is done for linear FEM. For that subject, there are numerous books and often lectures in engineering courses.

The sample results in this book, if not from table calculation, are mostly obtained with ANSYS, but other well-known FE codes use similar concepts such that the findings can be transferred.

This textbook describes the knowledge the author obtained over many years, the majority of them as a practical engineer. Most of it is common among experts. Therefore, the book does not list the origin of all these theories and algorithms but only gives advanced references. Since the book is derived from scripts of lectures, general solution methods are worked out in full when the related problem occurs for the first time.

This work is based on scripts of lectures being given by the author in the frame of Master's courses at Universities of Applied Sciences of Hanover (where the author is affiliated) and Lausitz as well as at the European School of Computer Aided Engineering Technology (ESoCAET). The roots, however, are teaching and development duties of the author during his long-lasting employment at CADFEM GmbH. The author would particularly like to thank its founder, Dr.-Ing. Günter Müller, for the opportunity to learn during everyday work as well as for his uncomplicated handling of possible copyright questions.

The author first earned his stripes in the field of Finite Elements—which already included a certain amount of non-linearity—at "Institut für Baumechanik und Numerische Mechanik" (Institute for Structural and Numerical Mechanics) of University of Hanover under the guidance of Prof. Dr.-Ing. Erwin Stein, who awakened the author's enthusiasm first for mechanics, then for Finite Elements and to whom the author gives his heartfelt thanks.

A German-language version of this book was first published in 2009.

Langenhagen, Germany — Wilhelm Rust
Spring 2014

Notation

Symbols of formulae are explained at least at their first appearance in the text.

M	Matrices are written in boldface and with capital letters
v	Vectors, row and column matrices in boldface and lower case letters, except a certain quantity is commonly noted in a different way
0	Means a zero vector or a zero matrix
I	A unit matrix (identity)
$\Delta(\ldots)$	Denotes an increment
$\tilde{a}$	a tilde over a variable—an approximation
$\bar{a}$	a bar—a given value
$\hat{a}$	a hat (circumflex)—a value associated to a Finite Element node
a^*	a star—a modified, improved value or one being used instead of the original one
FE	Finite Elements
FEM	Finite-Element Method
CoS	Coordinate system
eq.	Equation
s.o.eqs.	System of equations
deq.	Differential equation
r.h.s.	Right hand side
w.r.t.	With respect to
resp.	Respectively
1d, 2d, 3d	One-, two-, three-dimensional resp. the one-, two-, three-dimensional space
[...]	Points to the reference list

Contents

Chapter 1
Basic Mathematical Methods

This chapter occurs here because it is of relevance for all following sections. It is possible to skip it until the first applications are formulated.

1.1 Index Notation

As long as it is possible the matrix notation with the matrix product as the kernel is used in the governing formulae. If this is not sufficient to explain how the multiplication must be carried out the index notation is applied including the *sum convention*:

If an index appears in two factors of a product a sum must be formed, i.e. the summation symbol is left out. The sum is formed over the necessary length n, e.g. over the number of coordinate directions, over the number of nodes or the number of degrees of freedom:

$$C_{ik} = A_{ij}B_{jk} := \sum_{j=1}^{n} A_{ij}B_{jk} \qquad \text{means in matrix notation} \quad \mathbf{C} = \mathbf{AB} \tag{1.1}$$

Instead of transposition the other index is used for summation:

$$C_{ik} = A_{ji}B_{jk} \qquad \text{means in matrix notation} \quad \mathbf{C} = \mathbf{A}^T\mathbf{B} \tag{1.2}$$

Furthermore Kronecker's delta is used with

$$\delta_{ij} = \begin{cases} 1 & for \;\; i = j \\ 0 & otherwise \end{cases} \tag{1.3}$$

and the following rule

W. Rust, *Non-Linear Finite Element Analysis in Structural Mechanics*,
DOI 10.1007/978-3-319-13380-5_1

$$a_{ki}\delta_{ij} = a_{kj} \tag{1.4}$$

There is a sum over i but there is only a contribution if $i=j$.

In index notation only scalars are to be multiplied. Therefore the *order* of the factors can be changed. The summation—determining the order in matrix notation—is described by the indices which must not be changed.

1.2 Derivatives with respect to a Vector

Let $\mathbf{v}$ be a vector with the components v_i:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix} \tag{1.5}$$

If the derivative of a scalar a with respect to $\mathbf{v}$ is requested this means that derivatives w.r.t. each component must be formed and ordered in a row:

$$\frac{\partial a}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial a}{\partial v_1} & \frac{\partial a}{\partial v_2} & \frac{\partial a}{\partial v_3} & \cdots \end{bmatrix} \tag{1.6}$$

This order is necessary because the linearised variation of a is obtained by multiplying by the variation of $\mathbf{v}$:

$$\begin{aligned} \delta a &= \left[\frac{\partial a}{\partial v_1}\delta v_1 + \frac{\partial a}{\partial v_2}\delta v_2 + \frac{\partial a}{\partial v_3}\delta v_3 + \cdots\right] \\ &= \begin{bmatrix} \frac{\partial a}{\partial v_1} & \frac{\partial a}{\partial v_2} & \frac{\partial a}{\partial v_3} & \cdots \end{bmatrix} \begin{bmatrix} \delta v_1 \\ \delta v_2 \\ \delta v_3 \\ \vdots \end{bmatrix} = \frac{\partial a}{\partial \mathbf{v}}\delta \mathbf{v} \end{aligned} \tag{1.7}$$

The derivative of a (column) vector $\mathbf{a}$ w.r.t. $\mathbf{v}$ concerns all components of $\mathbf{a}$ such that a matrix is created:

$$\frac{\partial \mathbf{a}}{\partial \mathbf{v}} = \begin{bmatrix} \frac{\partial a_1}{\partial v_1} & \frac{\partial a_1}{\partial v_2} & \frac{\partial a_1}{\partial v_3} & \cdots \\ \frac{\partial a_2}{\partial v_1} & \frac{\partial a_2}{\partial v_2} & \frac{\partial a_2}{\partial v_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.8}$$

The following might not be commonly defined but is necessary at some sections of this book:

If such a matrix is transposed this is noted for the two vectors:

$$\left[\frac{\partial \mathbf{a}}{\partial \mathbf{v}}\right]^T = \begin{bmatrix} \frac{\partial a_1}{\partial v_1} & \frac{\partial a_2}{\partial v_1} & \frac{\partial a_3}{\partial v_1} & \cdots \\ \frac{\partial a_1}{\partial v_2} & \frac{\partial a_2}{\partial v_2} & \frac{\partial a_3}{\partial v_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} =: \frac{\partial \mathbf{a}^T}{\partial \mathbf{v}^T} \tag{1.9}$$

The second derivative of a scalar a w.r.t. $\mathbf{v}$ then becomes:

$$\frac{\partial^2 a}{\partial \mathbf{v}^T \partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\partial a}{\partial \mathbf{v}^T} = \frac{\partial}{\partial \mathbf{v}} \left[\frac{\partial a}{\partial \mathbf{v}}\right]^T = \begin{bmatrix} \frac{\partial^2 a}{\partial v_1 \partial v_1} & \frac{\partial^2 a}{\partial v_1 \partial v_2} & \frac{\partial^2 a}{\partial v_1 \partial v_3} & \cdots \\ \frac{\partial^2 a}{\partial v_2 \partial v_1} & \frac{\partial^2 a}{\partial v_2 \partial v_2} & \frac{\partial^2 a}{\partial v_2 \partial v_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.10}$$

This is always a symmetric matrix.

What is the derivative of a matrix $\mathbf{A}$ w.r.t. $\mathbf{v}$? This would be a hypermatrix, a three-dimensional matrix, which cannot be shown on a piece of paper (except writing one "plane" after the other). Let us look at index notation:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{v}} \quad \textit{means} \quad \frac{\partial A_{ij}}{\partial v_k} \tag{1.11}$$

(three indices). However, our final results are at most two-dimensional matrices. The derivatives (1.11) only occur if $\mathbf{A}$ is multiplied by a vector $\mathbf{w}$ before the derivative is carried out:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{v}} \mathbf{w} \quad \textit{means} \quad \frac{\partial A_{ij}}{\partial v_k} w_j \tag{1.12}$$

Then it is useful to calculated $\mathbf{Aw}$ first, getting a vector, and then to form the derivative, getting a (two-dimensional) matrix again. This is explained in detail by means of an example in Sect. 2.4.

1.3 Newton-Raphson Method

In the linear FEM a linear system of equations must be solved, e.g. by methods based on the Gaussian algorithm. A direct solution of a larger system of non-linear equations is usually impossible. Therefore in most cases the Newton- or Newton-Raphson scheme[1] is applied. It is well-known for one-dimensional non-linear equations.

The Newton-Raphson scheme is known for the determination of the roots of a function $f(x) = 0$. In case of a single variable the iteration formula reads:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1.13}$$

Simply written in a different notation:

$$x_{i+1} = x_i + \left(\left. \frac{df(x)}{dx} \right|_{x=x_i} \right)^{-1} (-f(x_i)) \tag{1.14}$$

There $i+1$ means the iteration step, thus the index 0 the initial value. For the n-dimensional problem $\mathbf{d}(\mathbf{u}) = \mathbf{0}$ (symbols from the disequilibrium forces $\mathbf{d}$ depending on displacements $\mathbf{u}$, see Sect. 2.3) this must be written as follows:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \underbrace{\left(\left. \frac{\partial \mathbf{d}(\mathbf{u})}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{u}_i} \right)}_{\mathbf{K}_T}^{-1} (-\mathbf{d}(\mathbf{u}_i)) = \mathbf{u}_i + \mathbf{K}_T{}^{-1}(-\mathbf{d}(\mathbf{u}_i)) \tag{1.15}$$

$\mathbf{K}_T$ is called *tangential matrix*, in conjunction with mechanical analyses *tangential stiffness matrix* as well. In mathematics it is also called *Jacobian* if it simply is the derivative of a vector $\mathbf{d}$ with respect to a vector $\mathbf{u}$ or *Hesseian* if the $\mathbf{d}$ is considered to be the derivative of a potential Π with respect to $\mathbf{u}$, thus $\mathbf{K}_T$ being the second derivative of Π.

In general it is unusual to form the inverse. A linear system of equations is solved instead. This leads to the following algorithm:

[1] Joseph Raphson was a contemporary of Isaac Newton and contributed significantly to the development of the method that is commonly known as Newton scheme. It is said that Thomas Simpson created the well known notation from above.