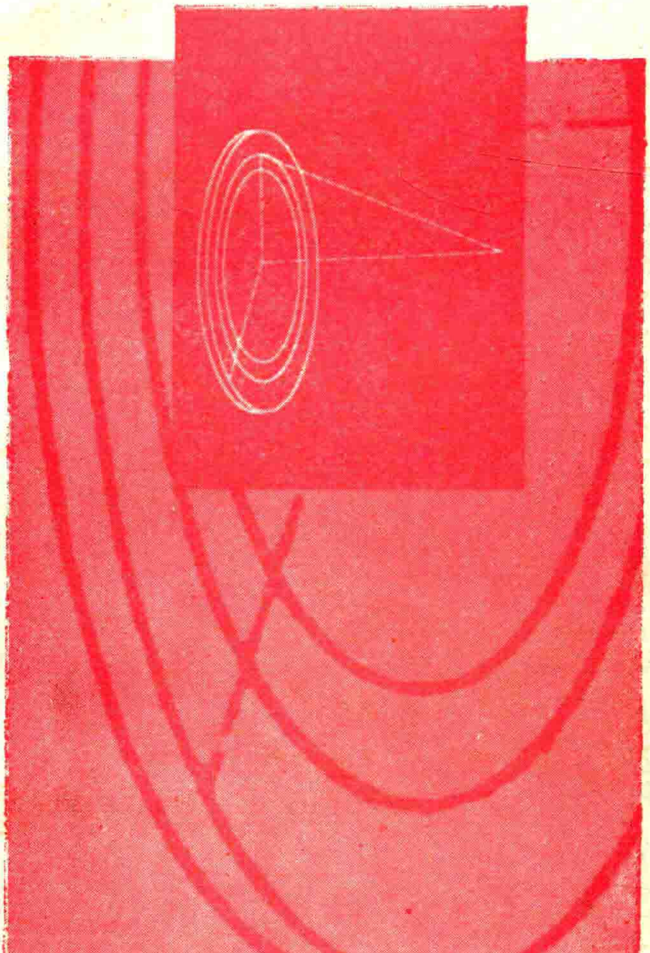
CALCULUS FOR PHYSICS



RICHARD DALVEN

CALCULUS FOR PHYSICS

Richard Dalven

Department of Physics
University of California, Berkeley

McGraw-Hill Book Company

New York St. Louis San Francisco Auckland Bogotá Hamburg
Johannesburg London Madrid Mexico Montreal New Delhi
Panama Paris São Paulo Singapore Sydney Tokyo Toronto

This book was set in Times Roman by Science Typographers, Inc.
The editor was Stephen Zlotnick.
The production supervisor was Diane Renda.
The cover was designed by Fern Logan.
Project supervision was done by The Total Book.
R. R. Donnelley & Sons Company was printer and binder.

CALCULUS FOR PHYSICS

1 2 3 4 5 6 7 8 9 0 DOCDOC 8 9 8 7 6 5 4

ISBN 0-07-015209-8

Library of Congress Cataloging in Publication Data

Dalven, Richard.
Calculus for physics.

Includes index.
1. Mathematical physics. 2. Calculus. I. Title.
QC20.D25 1984 530.1'55 83-25547
ISBN 0-07-015209-8

TO THE READER

The aim of this little book is to "bridge the gap" between the calculus you've learned in your mathematics classes and the calculus used in your physics courses. It has been my experience, based on several years of teaching physics, that many students who are reasonably familiar with the techniques of calculus are not familiar with the meanings of the derivative, differential, and integral when applied to physics. This book is an attempt to help such students and is an outgrowth of notes I distributed in my physics courses.

The book concentrates on explaining the meanings and uses of the key concepts of calculus as applied to elementary physics. However, its aim is not to teach physics per se, and the physics used is kept as simple as possible. The emphasis is on the derivative as a rate of change, the use of differentials as small quantities, and the integral as a sum, all in the context of physics. The book assumes you have taken, or are taking, a course in calculus, so it reviews the definitions and techniques of differentiation and integration, but does not attempt to teach them from the beginning. It is also assumed that you are familiar with elementary algebra and trigonometry, but a brief review of the latter is given in an appendix.

This book is essentially designed for self-study by a student who is beginning to learn physics. I would suggest that you work through the material fairly slowly, using your calculus book to refresh your memory, if necessary, on mathematical points. Most sections of the text conclude with a few exercises. These are designed to reinforce the concepts just presented and to give you practice in using them. The exercises are not meant to be a challenge and are both straightforward and moderate in number so you may realistically do them all. I suggest that you try the exercises as you work through the book. Detailed solutions (not just answers) to the exercises are given in the back of the book and should be consulted after you've given the problems a try.

I've made a real effort to explain the concepts clearly. In fact, you may sometimes think that I'm overexplaining. However, repetition is a useful tool in teaching, and I'd rather say too much about an important topic than say too little. In the same vein, I've tried to write in an informal tone, just as if I were lecturing to a small class. I've also kept this book short by concentrating on material I believe to be really important for physics. Calculus books today seem to run upwards of 1000 pages, possibly making them difficult to use as a reference or for self-study. I hope this little book will help you in your study of physics by making these important ideas more accessible to you.

The book has profited from the valuable comments of Ivanna Juricic, who also read the proofs with a meticulous eye, R. B. Hallock, T. D. MacIver, S. E. Rosser, and S. J. Shepherd, but the responsibility for errors is mine alone. It is again a pleasure to thank John Clarke for his generous hospitality at the Lawrence Berkeley Laboratory. The manuscript was typed with exceptional skill by Claudia Madison, to whom I extend sincere thanks for her help.

Richard Dalven

CONTENTS

CHAPTER
ONE

VARIABLES, FUNCTIONS, AND GRAPHS

INTRODUCTION

The aim of this chapter is to introduce much of the terminology we will use in the later chapters, and to review some of the basic concepts you have learned in your calculus course. These are variables, functions, and graphs. The point of view, however, will be that of the physicist, not the mathematician. The concepts we will discuss here are things we will use frequently as we progress.

VARIABLES AND FUNCTIONS

A *variable* is a quantity which may take on different values in the course of the discussion of some question. An example might be the radius r of a circle during a discussion of geometry. The idea of a variable should be contrasted with that of a *constant*, which is a quantity having a fixed value. Examples of constants are the numbers 6, 21, π, etc. The variable (or variables) under consideration in some situation will be denoted by appropriate and convenient symbols; an example is the symbol r for the radius of a circle.

Suppose x and y are both variables. A *function* is a rule connecting the two variables x and y such that, if the value of one variable (say x) is given, the value of the other variable is determined. For example, suppose

$$y = x^2 \tag{1.1}$$

Eq. (1.1) tells us that the value of the variable y is equal to the square of the value of the variable x. Eq. (1.1) therefore tells us that y is a function of x because, if we know the value of the variable x, the value of the variable y is determined. For example, if $x = 2$, $y = 4$, we say that Eq. (1.1) gives the variable y as a function of the variable x, or, more simply, Eq. (1.1) gives y as a function of x. In Eq. (1.1), assigning a value to x determines the value of y. We call x the *independent variable* in the function $y = x^2$ given in Eq. (1.1). The variable y is called the *dependent variable*, since the value of y depends on the value of x. In discussing Eq. (1.1), we say that y is a function of x, or that there exists a *functional relationship* between the variables y and x.

The function $y = x^2$ in Eq. (1.1) is such that only a single value of the dependent variable y corresponds to each value of the independent variable x. Such a function is called *single-valued*. Functions for which there are more than one value of the dependent variable for each value of the independent variable are called *many-valued* functions. An example of a many-valued function is $y = \pm\sqrt{x}$, in which there are two values of y for each value of x. When we indicate a square root, as in $\sqrt{x}$, or $(1 - x)^{1/2}$, we mean the positive square root; the negative square root would be indicated explicitly, as $-\sqrt{x}$, or $-(1 - x)^{1/2}$. Most of the functions we will encounter will be single-valued.

Given the existence of a functional relationship between the variables y and x, then the set of values which the independent variable x may take on is called the domain of the function. The set of values which the dependent variable y may take on is called the range of the function.

Consider a second example. The familiar relation between the area A of a circle and its radius r is given by Eq. (1.2). Eq. (1.2),

$$A = \pi r^2 \tag{1.2}$$

expresses the area A as a function of the radius r; π is a constant. From Eq. (1.2), if the value of the independent variable r is specified, then the value of the dependent variable A is determined according to the function (1.2) which gives A as a function of r.

In physics, the variables with which we deal are almost always quantities with a physical meaning and are things that can be measured. For example, the variables A and r in Eq. (1.2) are the area and radius of a circle, both of which are quantities with a physical meaning and which can be measured if we wish. We may contrast the case of Eq. (1.2) with that of Eq. (1.1), in which the variables x and y are mathematical symbols, whose physical meanings (if any) are not specified. In dealing with physical problems, it is helpful to keep in mind the meanings of the symbols with which we deal. In physics, we will constantly be working with functions which give a dependent variable of physical interest in terms of an independent variable (or variables) which will also be physically interesting and measurable quantities.

As a final example of a function, recall the familiar result from elementary physics that "distance equals rate times time" for a body moving with constant rate or speed. If the symbol s is used for distance, v for speed or rate, and t for time, our familiar result is expressed by

$$s = vt \tag{1.3}$$

Eq. (1.3) gives the distance s (the dependent variable) as a function of the time t (the independent variable) in the case in which the speed v is constant, so v is not a variable in this situation. Eq. (1.3) introduces us to a most important independent variable in physics—the time. Much of physics is concerned with how different quantities vary with time, so physics is often concerned with equations, like Eq. (1.3), giving some quantity *as a function of time*.

Exercises

1.1 Consider the variables w and u connected by the function

$$w = 7u^2 + 6u + 3$$

Which is the independent variable? Which is the dependent variable?

1.2 Eq. (1.1) above gives y as a function of x. Use Eq. (1.1) to obtain an equation giving x as a function of y. In the functional relation you obtained, identify the dependent and independent variables.

1.3 A familiar geometric relation is that between the circumference C and the radius r of a circle, which says that the circumference is the constant 2 times the constant π times the radius of the circle. Write the equation giving C as a function of r. In the functional relation between C and r, which is the dependent and which is the independent variable?

FUNCTIONAL NOTATION

We now discuss a few points concerning the notation used to express functions and functional relationships between variables.

The existence of a general functional relationship between two variables x and y may be indicated by writing

$$y = f(x) \tag{1.4}$$

an equation which tells us that the dependent variable y is some function f of the independent variable x, but we are not told what the specific function is. Eq. (1.4) does tell us, however, that y is a function of x, so y depends on x. If the specific function f is known, then that information may also be given. For example, in the example in Eq. (1.1) in which $y = x^2$, the specific function f is given, so we may write

$$y = f(x) = x^2 \tag{1.5}$$

Eq. (1.5) says that the dependent variable y is a function $f(x)$ of the independent variable x, and that the specific function $f(x)$ is x^2.

A situation often encountered in physics is the following. Suppose y is some function f of x, so

$$y = f(x) \tag{1.6}$$

and the variable x is itself a function g of another variable t, so

$$x = g(t) \tag{1.7}$$

Eq. (1.6) says y is a function $f(x)$ of the variable x; Eq. (1.7) says x is a function $g(t)$ of the variable t. One can combine Eqs. (1.6) and (1.7) by writing

$$y = f[g(t)] \tag{1.8}$$

Eq. (1.8) says that the variable y is a function f of the function $g(t)$ of the variable t, so the dependent variable y is ultimately a function of the independent variable t. One also describes the situation in Eq. (1.8) by saying that y is a function f of the function g of t.

As an example, suppose we have the relations

$$y = f(\theta) = \sin\theta \tag{1.9}$$

$$\theta = g(t) = \omega t \tag{1.10}$$

where, in Eq. (1.10), ω is a constant. Eq. (1.9) says that the variable y is a function (the sine) of the variable θ, while θ is a function of the variable t. We may combine Eqs. (1.9) and (1.10), using Eq. (1.8), to give

$$y = \sin \omega t \tag{1.11}$$

showing that y is a function of t.

Finally, one sometimes sees notation like the following. Suppose y is a function of x; one may write this functional relationship as

$$y = y(x) \tag{1.12}$$

simply to conserve symbols (which, surprisingly, are sometimes in short supply). Eq. (1.12) says that y is some (unspecified) function of x, so Eq. (1.12) conveys the same information as Eq. (1.6) but uses fewer symbols to do so. In physics, one frequently sees the relations like

$$x = x(t) \tag{1.13}$$

$$y = y(t) \tag{1.14}$$

saying that x is a function of t and y is a (different) function of t, so, in Eqs. (1.13) and (1.14), t is the independent variable in both equations.

FUNCTIONS OF SEVERAL VARIABLES

In the preceding sections, we discussed a function

$$y = f(x) \tag{1.15}$$

in which y depends on the single variable, x. The function $f(x)$ in Eq. (1.15) is a function of one independent variable. We may also consider functions of more than one variable, such as

$$w = g(x, y) \tag{1.16}$$

Eq. (1.16) says that the dependent variable w is a function of the two independent variables x and y; if values of x and y are specified, then the value of w is determined. Note that the two independent variables x and y are *independent of each other*, so x and y are separate independent variables.

An example of a function of two variables of the type given in Eq. (1.16) is

$$w = g(x, y) = x^2 + y^2 \tag{1.17}$$

A second example is the expression

$$V = \pi r^2 h \tag{1.18}$$

for the volume V of a right circular cylinder of radius r and height h.

In Eq. (1.18), V is a function of r and h, both of which are independent variables, so the volume V is a function $V(r, h)$ of r and h. Another example is the function

$$y = y(x, t) = A\sin(kx - \omega t) \tag{1.19}$$

where A, k, and ω are constants. The function in Eq. (1.19) gives y as a sinusoidal function of the two independent variables x and t.

While we will, in this book, usually restrict ourselves to dealing with functions of one variable, we will, on occasion, introduce functions of more than one variable.

VALUE OF A FUNCTION AT A POINT

Generally, the independent variable x in a functio..

$$y = f(x) \tag{1.20}$$

will take on a set of values called the domain of the function. What this set of values is will be determined by the particular function, the physical situation being discussed, etc. We will often wish to consider the value of the dependent variable y for some *particular* values of x. For example, if

$$y = f(x) = x^2 \tag{1.21}$$

then $y = 4$ when $x = 2$. We say the function $y = f(x) = x^2$ has the value 4 when $x = 2$, or, equivalently, that the function $y = 4$ at the point $x = 2$. (The use of the term *point* for a particular value of x refers to the point on the x axis, such as $x = 2$, corresponding to the value of x. Thus we refer to the value of a function at a point.)

If y is a function $f(x)$ of x, as in Eq. (1.20), then there is a common notation used to indicate the value of $f(x)$ for some particular value of x. It is usual to denote particular values of a variable by putting a subscript on the variable. Thus

$$x = x_1 \tag{1.22}$$

is an equation saying that the variable x has the particular value x_1. In the example above, we consider the case in which x has the value 2, so we were considering

$$x = x_1 = 2 \tag{1.23}$$

meaning that the variable x has the particular value $x_1 = 2$. If, at the same time, we wished to consider a different particular value of x, we might call it x_2. For example, we might consider

$$x = x_2 = 5 \tag{1.24}$$

as a second particular value of x.

If we are considering a function

$$y = f(x) \tag{1.25}$$

and if the independent variable x has the value x_1 so

$$x = x_1 \tag{1.26}$$

then we indicate the value of the function $f(x)$ when x has the value x_1 by the notation

$$f(x_1) \tag{1.27}$$

Expression (1.27) stands for the value of the function $f(x)$ when $x = x_1$, or, saying it in another way, the value of the function $f(x)$ at the point $x = x_1$. As an example, let's return to the function

$$y = f(x) = x^2 \tag{1.28}$$

and find the value of the function (1.28) when x has the value 2, i.e., when

$$x = x_1 = 2 \tag{1.29}$$

Using the notation described above, we have

$$f(x_1) = (x_1)^2 = (2)^2 = 4 \tag{1.30}$$

Since $x_1 = 2$, we can (and will often) also write

$$f(2) \tag{1.31}$$

for the value of the function $f(x)$ when $x = 2$. Similarly, when $x = 5$, $f(5)$ is the value of the function $f(x)$ when $x = 5$.

To summarize, if y is a function $f(x)$ of x, so $y = f(x)$, then $f(x_1)$ is the value of the dependent variable y, and of the function f, when the independent variable x has the value x_1.

These ideas may be extended to a function of several variables. Suppose

$$w = f(x, y) \tag{1.32}$$

is a function of the two variables x and y. Then

$$f(x_1, y_1)$$

is the value of the function f when x has the value x_1 (i.e., $x = x_1$) and y has the value y_1 (i.e., $y = y_1$). For example, suppose

$$w = f(x, y) = x^2 + y^2 \tag{1.33}$$

What is the value of $f(0,2)$, which is the value of the function f when $x = 0$ and $y = 2$? From Eq. (1.33),

$$f(0,2) = (0)^2 + (2)^2 = 4 \tag{1.34}$$

Note also that we may sometimes wish to consider a function like $f(x, y)$ for a particular value of one independent variable but for *any* value of the other independent variable. As an example consider the function f in Eq. (1.33) when $x = 0$; then we have

$$f(0, y) = y^2 \tag{1.35}$$

as the value of $f(x, y)$ when $x = 0$. In this case the "value" of $f(0, y)$ is itself a function (in this case the function y^2) and not just a number.

Exercises

1.4 Given the function

$$y = f(x) = 3x^2 + 2$$

Find (a) $f(2)$; (b) $f(0)$; (c) $f(-1)$.

1.5 Given the function

$$y(x, t) = A \sin(kx - \omega t)$$

where A, k, and ω are constants. Find expressions for: (a) $y(0,0)$; (b) $y(0, t)$; (c) $y(x,0)$

1.6 Given the function $y = f(x) = (1 - x^2)^{1/2}$. Find $f(x - a)$, where a is a constant. Note that $f(x - a)$ is obtained by replacing the independent variable x in $f(x)$ by the new variable $(x - a)$.

THE GRAPH OF A FUNCTION

When we are considering a function, it is usual in physics to display the functional relationship between the dependent and independent variables by means of a *graph* of the function. Suppose we are considering a function

$$y = f(x) \tag{1.36}$$

We can assign a series of values to the independent variable x, and, from the functional relationship in Eq. (1.36), obtain the corresponding values of the dependent variable y. For example, if the specific function under discussion were

$$y = f(x) = x^2 \tag{1.37}$$

we would obtain the table of values below for integral

x	0	1	2	3	4	5
y	0	1	4	9	16	25

values of x between 0 and 5. However, it is difficult from a table of values to "see" the behavior of the function $f(x)$ as x takes on various values. For this reason it is usual to construct a graph of the function.

The definition of a graph is as follows. The graph of the function $f(x)$ is the set of all points with rectangular coordinates $(x, f(x))$. Here the notation $(x, f(x))$ means the point whose abscissa (x coordinate) has the value x and whose ordinate (y coordinate) has the value $f(x)$. For our example in Eq. (1.37), the graph of the function $y = f(x) = x^2$ is the set of all points with coordinates (x, x^2). Six such points, (0, 0), (1, 1), (2, 4), (3, 9), (4, 16), and (5, 25) are seen in the table above. These six points are, of course, not *all* of the points comprising the graph of the function $y = f(x) = x^2$; there are an infinite number of other points, such as (2.5, 6.25), with the form (x, x^2).

If we plot, in two dimensions, all of the points of the form $(x, f(x))$, we obtain the graph of $f(x)$. For our example, we plot points of the form (x, x^2) and obtain the graph shown in Fig. 1.1. In that figure, we plot $y = f(x) = x^2$ vertically (ordinate) and x horizontally (abscissa) and obtain the curve shown in the figure. The graph is the set of all points whose coordinates (x, y) satisfy the functional

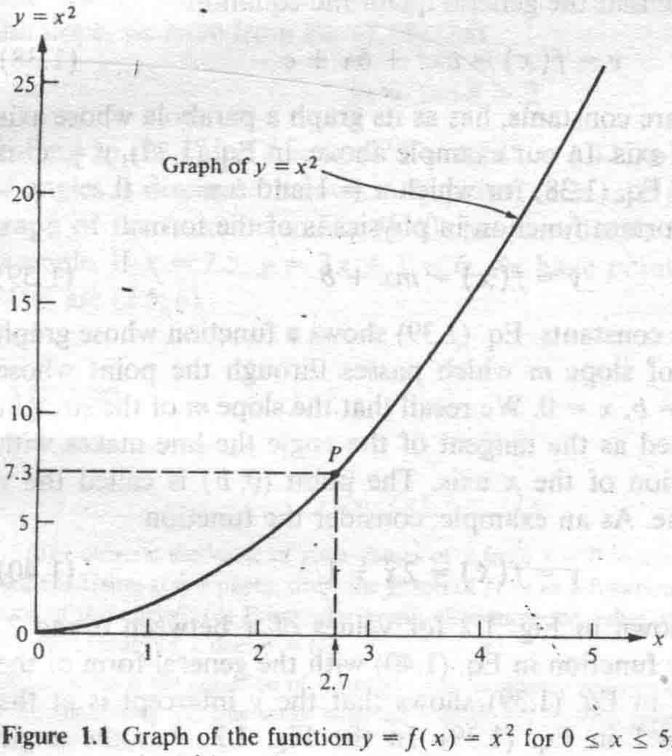


Figure 1.1 Graph of the function $y = f(x) = x^2$ for $0 \leq x \leq 5$.

relation $y = x^2$ given in Eq. (1.37). Therefore, any point on the graph (or curve, as it is also called) has coordinates (x, y) such that $y = x^2$. The point P, for example, has coordinates (2.7, 7.29). From the figure, we can "see" how the function $y = f(x) = x^2$ varies with x.

In general, then, if we have the graph of the function $y = f(x)$ plotted as a function of x, we may find the value of the function for any value $x = x_1$ of x by reading it off the graph. The point $(x_1, f(x_1))$ will be a point on the graph, so, knowing the value of x_1 determines the value of $f(x_1)$. In Fig. 1.1, in which $f(x) = x^2$, we considered the point $x_1 = 2.7$ on the x axis; the y coordinate of the point P can be read to be $f(x_1) \cong 7.3$, as shown in the curve in the figure. Note that we can read the value of $f(x_1) = f(2.7)$ off the graph in Fig. 1.1 only approximately. From the graph, we read $f(2.7) \cong 7.3$, which may be compared with the exact value $f(2.7) = (2.7)^2 = 7.29$. (It should be pointed out that, in physics, the reading of graphs in this manner is frequently necessary for graphs of experimental data for which the functional relationship is not known. In such cases, the reading of values is necessarily approximate.)

There are a number of functions of one variable whose graphs are frequently encountered in physics. One is the parabola, an example of which is seen in the graph of $y = x^2$ in Fig. 1.1. From analytic geometry, we know that the general quadratic equation

$$y = f(x) = ax^2 + bx + c \tag{1.38}$$

where a, b, and c are constants, has as its graph a parabola whose axis is parallel to the y axis. In our example above, in Eq. (1.37), $y = x^2$ is the special case of Eq. (1.38) for which $a = 1$ and $b = c = 0$.

A second important function in physics is of the form

$$y = f(x) = mx + b \tag{1.39}$$

where m and b are constants. Eq. (1.39) shows a function whose graph is a straight line of slope m which passes through the point whose coordinates are $y = b$, $x = 0$. We recall that the slope m of the straight line (1.39) is defined as the tangent of the angle the line makes with the positive direction of the x axis. The point $(0, b)$ is called the y intercept of the line. As an example, consider the function

$$y = f(x) = 2x + 1 \tag{1.40}$$

whose graph is shown in Fig. 1.2 for values of x between 0 and 3. Comparison of the function in Eq. (1.40) with the general form of the straight line given in Eq. (1.39) shows that the y intercept is at the point (0, 1) so $b = 1$ in Eq. (1.39). In the Fig. 1.2, θ is the angle