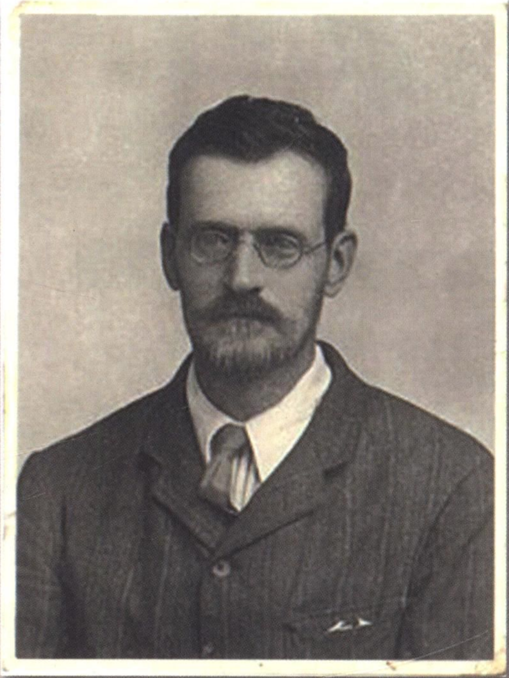
An Invitation to the ROGERS-RAMANUJAN IDENTITIES



Andrew V. Sills

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Mathematics

The Rogers–Ramanujan identities are a pair of infinite series—infinite product identities that were first discovered in 1894. Over the past several decades, these identities, and identities of similar type, have found applications in number theory, combinatorics, Lie algebra and vertex operator algebra theory, physics (especially statistical mechanics), and computer science (especially algorithmic proof theory). Presented clearly and coherently, **An Invitation to the Rogers–Ramanujan Identities** is the first book entirely devoted to the Rogers--Ramanujan identities and includes related historical material that is unavailable elsewhere.

Features

- The first book focused entirely on the Rogers–Ramanujan identities.
- Prerequisites kept to a minimum, although a some mathematical fluency and sophistication will be required.
- Material is presented in a generally historical order, but author does not hesitate to (anachronistically) introduce modern methods as necessary, when doing so will greatly simplify the presentation.
- Previously unpublished primary source historical material is included where appropriate.

Andrew Sills earned his PhD in 2002, from the University of Kentucky, under George E. Andrews, Evan Pugh Professor of Mathematics, Pennsylvania State University. He was Hill Assistant Professor of Mathematics at Rutgers University from 2003–2007 and a tenure-track Assistant Professor at Georgia Southern University between 2007-2011. In 2011, he was promoted Associate Professor at Georgia Southern, and became a full Professor of Mathematics, in 2015. He is a permanent Member of DIMACS (Center for Discrete Mathematics and Computer Science), since 2011. His research during 2014-2015, was partially supported by a grant "Computer Assisted Research in Additive and Combinatorial Number Theory and Allied Areas" funded by the National Security Agency.

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The only known photograph of L. J. Rogers appears on the cover and on page xv. This photo appeared, undated and uncredited, in Rogers' Royal Society obituary in December 1934.

The image of S. Ramanujan appearing on the cover and on page xvi is from one of four extant photographs of Ramanujan according to Bruce Berndt, "The Four Photographs of Ramanujan," *Ramanujan: Essays and Surveys*, ed. B. C. Berndt and R. A. Rankin, in: *History of Mathematics*, vol. 22, American Mathematical Society and London Mathematical Society, 2001. The photographer is unknown, but according to Berndt, the photograph was likely taken during the summer of 1916.

The publisher believes both images to be in the public domain.

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An Invitation to the ROGERS-RAMANUJAN IDENTITIES

To my family

Foreword

I first met the Rogers–Ramanujan identities in the fall semester of 1961 at the University of Pennsylvania. The course was taught by Hans Rademacher, one of the great number theorists of the twentieth century. He clearly was in agreement with Hardy's comment that "[i]t would be difficult to find more beautiful formulæ than the 'Rogers–Ramanujan' identities" [Ram27, p. xxxiv]. Rather poignantly, he believed that D. H. Lehmer had proved that these two identities were an isolated phenomenon. While this belief later turned out to be false, it is clear that Rademacher found these results to be so beautiful that it was important to reveal such esthetic excellence to beginning graduate students.

Rademacher's pessimistic view of the Rogers–Ramanujan identities as beautiful but singular results turned out to be far from the truth.

The recognition of the Rogers–Ramanujan identities as the tip of a research cornucopia is surely attributable to Basil Gordon's generalization in 1961 [Gor61]. Rademacher was completely unaware of Gordon's paper, and, consequently, so were all of his students (including me). I would note that in "Some Debts I Owe" [And01], I mention how Rademacher's proof of the Rogers–Ramanujan identities led me to an independent discovery of Gordon's theorem. I was truly deflated when I learned of Gordon's work months after I had submitted this grand generalization for publication.

To make a long story short, I was completely captivated by the Rogers–Ramanujun identities after Rademacher's enticing reënactment of Schur's proof. Since then they have been a constant theme in much of my research.

Now Andrew (Drew) Sills has put together this great introduction to these fascinating results. Drew was my student and wrote his PhD thesis under my direction, elucidating and extending the work of Lucy Slater on a variety of q-series identities of Rogers–Ramanujan type.

This is a marvelous book. Drew has drawn on his encyclopedic knowledge of the literature to prepare this coherent and exciting account of the Rogers–Ramanujan identities and the aftermath. The book has numerous exercises and so could well be used as a text in a graduate course. Drew has clearly in mind that this book is an introduction. So, more advanced topics are saved to the final "But wait... there's more" chapter where pointers are given to sources in the literature.

But wait... there's even more! Appendix A provides an extended version of Lucy Slater's list, plus an extended list of false theta function identities, and Appendix B contains some of the wonderful letters written by the early pioneers of the subject.

This is wonderful. Thank you, Drew, for this excellent book!

George E. Andrews
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Preface

Leonard James Rogers (1862–1933)

In 1894, a relatively unknown English mathematician at Yorkshire College named Leonard James Rogers published a paper entitled "Second memoir on the expansion of certain infinite products," in the *Proceedings of the London Mathematical Society* [Rog94]. This second installment of a three-part series on the expansion of infinite products was 26 pages long. Buried on the tenth page of this paper, and written in obscure notation, is an identity of analytic functions, valid when $|q| < 1$, that later came to be known as the first Rogers–Ramanujan identity [Rog94, p. 328]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \tag{0.1}$$

What we now know as the second Rogers–Ramanujan identity occurs two pages later [Rog94, p. 330]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \tag{0.2}$$

G. H. Hardy's account of how (0.1) and (0.2) were discovered by Rogers, ignored by the mathematical community, only to be rediscovered (conjecturally) many years later by Srinivasa Ramanujan is quoted in Chapter 2.

Srinivasa Ramanujan (1887–1920)

The uninitiated will naturally wonder how any author could purport to write an entire book about nothing more than a pair of identities of analytic functions. In contrast, experts in the field will realize immediately upon picking up this volume that it cannot possibly be long enough to adequately discuss the multitude of subjects arising from (0.1) and (0.2). In the interest of full disclosure, yes, this volume is but an *introduction* to *certain aspects* of the Rogers–Ramanujan identities, and an invitation to study further.

The Rogers–Ramanujan identities, and identities of similar type, arise in number theory, analysis, combinatorics, the theory of integer partitions, vertex operator algebras, representation theory of Lie algebras, statistical mechanics, and knot theory. A goal of this book is to present and develop the Rogers–Ramanujan identities from the perspective of the theory of basic hypergeometric series and the theory of integer partitions. Historical material, in the form of remarks, and in some cases longer quotes and explanations, is woven into the narrative. While I have in some sense attempted, in broad terms, to present the material in a kind of "historical arc," I have, for mathematical efficiency and expediency, introduced modern techniques as required. For example, I begin with a "prehistory" chapter where integer partitions and hypergeometric series are introduced in order to set the stage for the unveiling of the Rogers–Ramanujan identities in the next chapter. However, in order to deal with hypergeometric series and their generalizations efficiently, I introduce the techniques of Wilf and Zeilberger, which of course made their debut nearly a hundred years *after* the discovery of the Rogers–Ramanujan identities. This is done so for two reasons: to convey how modern practitioners approach (q-)hypergeometric series, and so that a plethora of useful classical results can be dispensed with via one-line proofs.

In the course of this work, I take the liberty of not attempting to make the narrative entirely "self-contained" in the sense that I mention related topics and results of interest, even if I do not have the space to fully develop the topic and give proofs of those results. To this end, one of the purposes of this book is to alert interested readers to what related material is in the literature and guide them to that literature. A case in point is the Rademacher convergent series for the unrestricted partition function, and related results for various restricted partition functions. A number of my students over the years were instantly enthralled with these theorems and were led to study the circle method as a result. Accordingly, I mention them in this book, but as excellent expositions of the circle method are readily available elsewhere in the literature, I direct the readers to those sources rather than providing a similar development here.

In another attempt to engage readers who may be interested in the lives of some of the people who are responsible for the mathematics discussed in this book, I have included in an appendix transcriptions of letters written by W. N. Bailey to Freeman Dyson and to Lucy Slater. Readers will gain some "inside" information on how the mathematics is actually *done*, before it was ready to be written up as the polished final product that we read in the journals.

The mathematical prerequisites assumed are fairly minor. In some sense, not much beyond an elementary calculus course and an introductory number theory course is required to follow the statements and proofs of the theorems included. On the other hand, a certain level of mathematical experience and sophistication is required to fully appreciate the material presented. For instance, I have found that in most cases, students who have recently completed a year of calculus, and thus have some experience with power series, are nonetheless not quite ready to work with generating functions. The first two chapters are long and contain many exercises. These two chapters alone could easily form the material of a special topics graduate course. Chapters 3–5 contain more advanced material, and fewer explicitly stated exercises, as graduate students and practitioners at this level will have the sophistication to ask themselves relevant questions, and practice as needed to master the material to their own satisfaction. Chapter 6 contains eight sections, each of which, if more fully developed, could have been a chapter in its own right. The interested reader is pointed to the literature for more information.

All that having been stated, I hope that there is plenty contained in this volume to engage both graduate students and strong undergraduates, as well as material that professional mathematicians and scientists will find both useful and delightful.

Andrew Sills
Savannah, Georgia
June 2017

Acknowledgments

I have many people to thank for helping this book come into being.

In particular, this project began as a result of Sarfraz Kahn of CRC Press asking to meet with me at the Joint Meetings of the American Mathematical Society and the Mathematical Association of America held in San Antonio, Texas, in January, 2015. He has been supportive and encouraging throughout the entire process. I also wish to express my gratitude to Project Editor Robin Lloyd-Starkes and her team for all their help, and to "TeX-pert" Shashi Kumar for assistance with typesetting issues.

I thank George Andrews for being my teacher, thesis advisor, mentor, and friend over many years. George first introduced me to the Rogers–Ramanujan identities, and clearly this book would have never come into being without his guidance and inspiring influence. I also thank George for granting me access to his files, including the letters W. N. Bailey wrote to Lucy Slater. I am also grateful for his writing the *Foreword* to this book.

I am grateful to Freeman Dyson for preserving the letters he received from W. N. Bailey in the 1940s and for allowing me to transcribe and present them in this book.

I thank Mike Hirschhorn for much stimulating conversation and correspondence over the years, and in particular for pointing out that Marshall Hall's notes on Hardy's 1936 IAS lectures [Har37b] contain a statement that Ramanujan had in fact found a proof of the Rogers–Ramanujan identities in 1917 shortly before Ramanujan stumbled upon Rogers' published proof in the *Proceedings of the London Mathematical Society.*

Thanks are due to Jim Lepowsky, who read an early version of the manuscript and patiently shared his mathematical insights and helped with the wording of a number of sections, especially those which dealt with the intersection of Rogers–Ramanujan type identities with vertex operator algebra theory. Any misstatements or questionable wording that remain are entirely the fault of the author, and remain despite Jim's painstaking efforts.

I thank Ken Ono for sharing thoughts on how best to present the material in §6.5.

Many thanks to Jimmy McLaughlin and Peter Zimmer for collaborating with me on a number of research projects over the years, some of which are mentioned in this book.

I extend my sincere gratitude to Robert Schneider, who graciously volunteered to proofread the entire book before it went to press, and made numerous helpful suggestions.

I appreciate Doron Zeilberger for mentoring me during my postdoctoral years, and for continuing to encourage and collaborate with me since that time. Doron's influence will be felt throughout the book. In addition to the obvious use of his algorithms for efficiently proving hypergeometric and basic hypergeometric identities, his preference that the omnipresent variable q be treated as a formal variable in every instance when it is possible to do so, has been honored.

Others that certainly need to be acknowledged include Krishnaswami Alladi, Richard Askey, Alex Berkovich, Bruce Berndt, Doug Bowman, David Bressoud, Stefano Capparelli, Yuriy Choliy, Sylvie Corteel, Dennis Eichhorn, Frank Garvan, Ira Gessel, Mourad Ismail, Shashank Kanade, Karl Mahlburg, Steve Milne, Debajyoti Nandi, Peter Paule, Helmut Prodinger, Matthew Russell, José Plíno de Oliveira Santos, Carla Savage, James Sellers, Nic Smoot, Michael Somos, Dennis Stanton, Ole Warnaar, the late Herbert S. Wilf, Robert Wilson, and Ae Ja Yee.

And, of course, I thank my wonderful family for being my support network and my *raison d'être*.