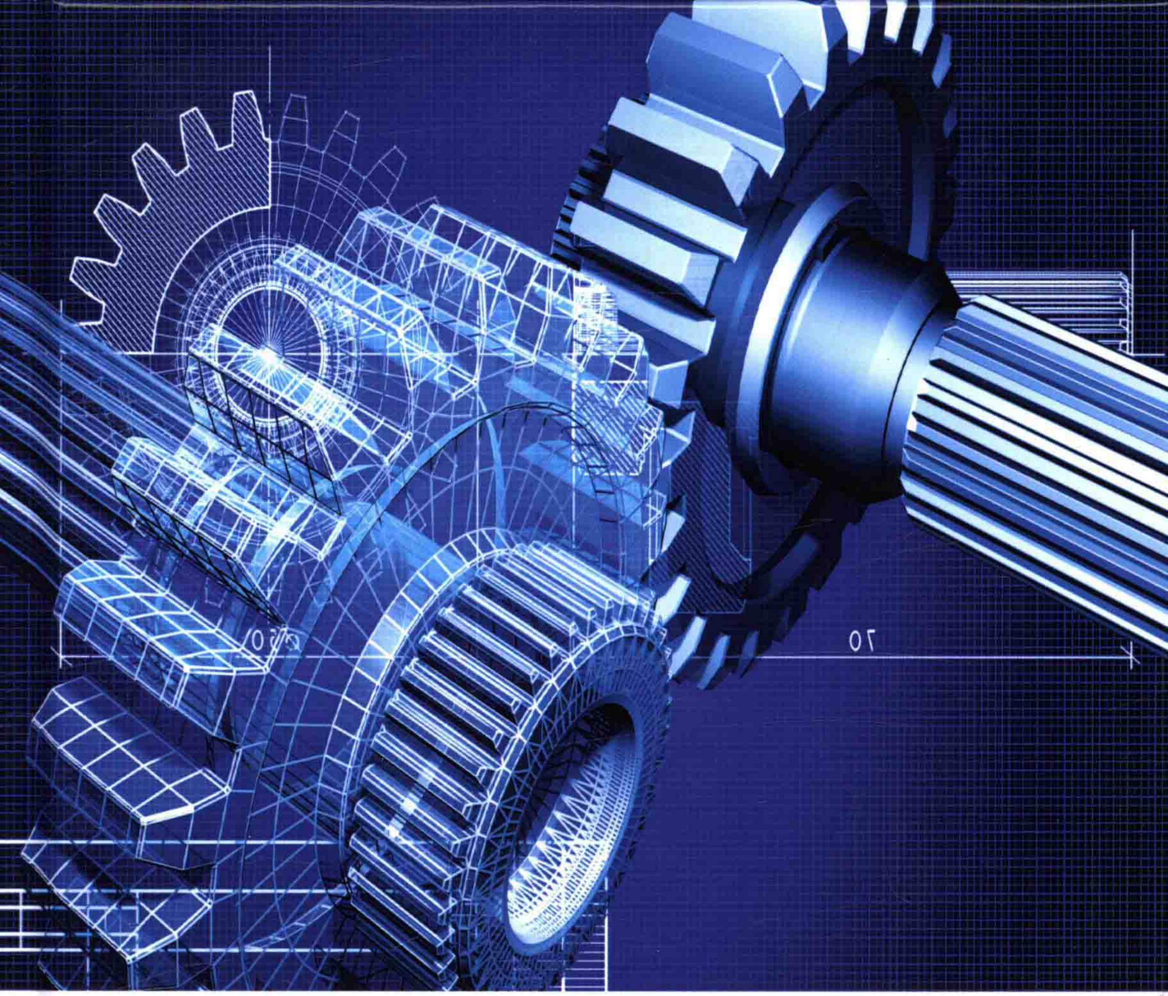

Multi-Body Kinematics and Dynamics with Lie Groups

Dominique P. Chevallier
Jean Lerbet

ISTE PRESS

ELSEVIER

The book explores the use of Lie groups in the kinematics and dynamics of rigid body systems.

The first chapter reveals the formal properties of Lie groups on the examples of rotation and Euclidean displacement groups. Chapters 2 and 3 show the specific algebraic properties of the displacement group, explaining why dual numbers play a role in kinematics (in the so-called screw theory). Chapters 4 to 7 make use of those mathematical tools to expound the kinematics of rigid body systems and in particular the kinematics of open and closed kinematical chains. A complete classification of their singularities demonstrates the efficiency of the method.

Dynamics of multibody systems leads to very big computations. Chapter 8 shows how Lie groups make it possible to put them in the most compact possible form, useful for the design of software, and expands the example of tree-structured systems.

This book is accessible to all interested readers as no previous knowledge of the general theory is required.

Dominique P. Chevallier is Emeritus Research Director at Navier Laboratory, Ecole Nationale des Ponts et Chaussées in France. His research interests are the mathematical methods of mechanics.

Jean Lerbet is Professor at the Universities of Evry and Paris Saclay in France. His research interests include differential geometry and mechanics, and stability.

ELSEVIER

ISTE PRESS

www.iste.co.uk

9781785482311

Dominique P. Chevallier
Jean Lerbet

Multi-Body Kinematics and Dynamics with Lie Groups

ISTE PRESS
ELSEVIER

Series Editor
Noël Challamel

Multi-Body Kinematics and Dynamics with Lie Groups

Dominique P. Chevallier
Jean Lerbet

ELSEVIER

First published 2018 in Great Britain and the United States by ISTE Press Ltd and Elsevier Ltd

ISTE Press Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

Elsevier Ltd
The Boulevard, Langford Lane
Kidlington, Oxford, OX5 1GB
UK

www.elsevier.com

Notices
Knowledge and best practice in this field are constantly changing. As new research and experience broaden our understanding, changes in research methods, professional practices, or medical treatment may become necessary.

Practitioners and researchers must always rely on their own experience and knowledge in evaluating and using any information, methods, compounds, or experiments described herein. In using such information or methods they should be mindful of their own safety and the safety of others, including parties for whom they have a professional responsibility.

To the fullest extent of the law, neither the Publisher nor the authors, contributors, or editors, assume any liability for any injury and/or damage to persons or property as a matter of products liability, negligence or otherwise, or from any use or operation of any methods, products, instructions, or ideas contained in the material herein.

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
Library of Congress Cataloging in Publication Data
A catalog record for this book is available from the Library of Congress
ISBN 978-1-78548-231-1

Multi-Body Kinematics and Dynamics with Lie Groups

List of Notations

$\mathbb{N}$:	set of integers
$\mathbb{Z}$:	ring of rational integers
$\mathbb{R}$:	real number field
$\mathbb{H}$:	real quaternion field (Appendix 1 to Chapter 1)
$\boldsymbol{\Delta}$:	dual number ring (see section 2.1.1)
ϵ:	dual number such that $\epsilon^2 = 0$ (see sections 1.5.6 and 2.1.1)
Df:	differential of the map f (see section 1.1)
f^T:	tangent map of the map f (see section 1.1)
$\circ$:	symbol of the composition of maps
$\mathcal{E}$:	Euclidean affine space (generally of dimension 3)
$\mathrm{Tr}\,(\mathbf{u})$:	translation in $\mathcal{E}$ by the vector $\mathbf{u} \in \mathbb{E}$ (see section 1.5.3)
$\mathbb{E}$:	vector space (very often the Euclidean of dimension 3 over $\mathbb{R}$)
$\wedge$:	vector product (or "cross product") in the oriented dimension 3 Euclidean vector space
$\tilde{\mathbf{a}}$:	mapping $\mathbf{x} \mapsto \mathbf{a} \wedge \mathbf{x}$ in the oriented dimension 3 Euclidean vector space
$[\cdot \mid \cdot]$:	Lie bracket in a Lie algebra
$(\cdot;\cdot;\cdot)$:	mixed product in the dimension 3 Euclidean vector space
$\{\cdot \mid \cdot\}$:	dual mixed product in the $\boldsymbol{\Delta}$-module $\mathbb{D}$ (see section 2.1.4)
$\{\cdot;\cdot;\cdot\}$:	dual mixed product in the $\boldsymbol{\Delta}$-module $\mathbb{D}$ (see section 2.1.4)
$\mathcal{L}(\mathbb{E})$:	algebra of the linear operators in the vector space $\mathbb{E}$ (see section 1.2.1)
$\mathrm{Gl}(\mathbb{E})$:	group of the regular linear operators the vector space $\mathbb{E}$ (see section 1.2.1)
$\mathrm{O}(\mathbb{E})$:	Orthogonal group of the Euclidean vector space $\mathbb{E}$ (see section 1.2)
$\mathrm{SO}(\mathbb{E})$:	Special othogonal group of the Euclidean vector space $\mathbb{E}$ (see section 1.2)
$\mathbb{U}$:	group of the normalized quaternions
$\mathrm{Ga}(\mathcal{E})$:	group of affine transformations of $\mathcal{E}$

$\mathbb{D}$, $\mathbb{D}(\mathcal{E})$: Displacement group of $\mathcal{E}$

$\mathfrak{D}$, $\mathfrak{D}(\mathcal{E})$: Lie algebra of $\mathbb{D}$ (of the skewsymmetric vectorfields on $\mathcal{E}$ see section 1.5.4)

$[\![\cdot \mid \cdot]\!]$: Klein form of $\mathfrak{D}$ (see section 1.5.5)

$\mathfrak{Z}$: set of vector fields $\in \mathfrak{D} - \mathfrak{T}$ vanishing on their axe (see section 1.5.5)

$\mathfrak{Z}_a$: set of vector fields $\in \mathfrak{D} - \mathfrak{T}$ vanishing at $a \in \mathcal{E}$ (see section 1.5.5)

$\mathfrak{T}$: ideal of $\mathfrak{D}$ of the constant vector fields on $\mathcal{E}$ (see section 1.5.4).

Δ_X: axis of $X \in \mathfrak{D}$ (see section 1.5.4)

f_X or p_X: pitch of $X \in \mathfrak{D}$ (see section 1.5.4)

$\mathbb{G}$: Lie group (defined according to the context)

$T\mathbb{G}$: tangent space of $\mathbb{G}$

$T_g\mathbb{G}$: tangent vector space of $\mathbb{G}$ at g

$\mathfrak{g}$: Lie algebra of the Lie group $\mathbb{G}$

Ad: adjoint representation of a Lie group (defined according to the context, see sections 1.2.2, 1.5.1,1.5.4)

A_*: equivalent to $\mathrm{Ad}\,A$ in the Euclidean displacement group (see section 1.5.2)

ad: adjoint representation of a Lie algebra (defined according to the context, see section 1.2.2)

ϑ_ℓ, ϑ_r: Left and right Maurer-Cartan forms on a Lie group (generally $\mathrm{SO}(\mathbb{E})$ or $\mathbb{D}$ see sections 1.2.2 or 1.5.1)

Introduction

The first significant occurrence of Lie groups in classical mechanics is due to V. Arnold in the paper [ARN 66] (1966) who studies Eulers's equations for the dynamics of a rigid body or of a perfect fluid and points out that, up to the choice of the group, their structure is similar. Today, articles of mechanics and physics referring to Lie groups, especially in Hamiltonian dynamics or in control theory are various and very numerous. However they often focus on theoretical properties, such as integrability or reduction with the help of first integrals, of rather particular mechanical systems which are of little interest for the common engineer who encounters complicated mechanical systems and wishes to simulate their behavior with a computer. In other words, the presentation of the dynamics of a single rigid body in the light of Lie group theory is more or less classical but limited in scope so that extensions to large mechanical systems, falling in the scope of mechanical engineering, are not quite common. Certainly many articles in applied mechanics refer to Lie groups, for instance in their title, but indeed they often make no real use of the powerful mathematical techniques of algebra and differential calculus derived from the structure of Lie group as it is understood in mathematics.

There are various methods to describe the configurations of rigid body systems with coordinates.Indeed they all amount to describe by one or another technique Euclidean displacements performed by the elements of the system and the significant mechanical properties are those which, after all, can be expressed in the language of this group. The goal of this book is double: first to show that the concept of Lie group can be useful to mechanical engineering, second to show that the calculations with Lie groups are powerful, very easy to handle in practical mechanical problems, on one condition: to make a small effort in order to learn some rules. The book aims at demonstrating that those required rules are not numerous, and that they make a complete system to state all the problems of general mechanics in a very compact form fully compatible with numerical or algebraic softwares. Whereas the common approach to the modeling of a complex mechanical system starts with a "forest of frames", the modelization based on Lie groups needs no necessary frame and no coordinates and is expressed in intrinsic form translatable in computer language.

Concerning the first objective we may remark that, in multibody mechanics, the efficiency of Lie group calculations, in the true sense, were soon demonstrated by applications (see [MIZ 92, MIZ 88, MON 84], going with the theoretical works [LER 88, CHE 86]). The task will be to extend the Lie group language from the more or less classical applications to the mechanics of a single rigid body to systems of rigid bodies and the above mentioned rules contain all the necessary algebraic and differential calculus required to generate their kinematic and dynamic equations. In the model of the systems we consider in the book we assume that the joints may be described Lie subgroups of the Euclidean displacement group what is the most common occurrence. Of course for final numerical calculations it will be necessary to refer to some coordinate system but a big advantage of the method is that the heavy calculations may be switched to the computer; only the mathematical structure of the statement of the mechanical problem in the language of differential calculus in Lie groups will be necessary at the preliminary stages of the design of a software.

The second objective is perhaps the leading motivation of this work. It was mentioned above: as soon as they concern rigid bodies, the calculations in kinematics and dynamics are calculations the Euclidean group $\mathbb{D}$ in dimension 3 which is a classical Lie group. From this standpoint they may be roughly distributed among three main levels according to the sharpness of the mathematical structure which is concerned at each level.

– The calculations of level 1, do not refer to the particular features of the Euclidean group, they only rely on the general Lie group structure of $\mathbb{D}$ (that is to say only the algebraic structure of group and the structure of manifold allowing the differential calculus).

– The calculations of level 2, use a particular feature of the Euclidean displacement group and they refer to the splittings of $\mathbb{D}$ into a "translation group" and a "rotation group" about some fixed origin in space.

– The calculations of level 3 are performed in the coordinate language where all the quantities are described by matrices depending on coordinates in $\mathbb{D}$ (as Euler angles or Cayley-Klein parameters and so on).

It is at level 2 that, in mechanics, all relation splits into a "linear part" and an "angular part", that the velocity of a rigid object may be described by a linear and an angular velocity, that a "torsor" splits into two "Plücker's vectors". At this level all the calculations may be performed in dimension 3 with standard vector algebra, but the structure of all formulas are much more complicated than at level 1.

At level 1, the calculations may be performed with a well defined algebra in dimension 6[1]. It is at this level that all the mathematical relations take the most compact form so that it is worth to perform the most possible part of calculations at this level. When it will be necessary to come back to a more familiar mathematical formalism the translation of those relations into (more complicated!) relations taking into account properties of the Euclidean group and Euclidean geometry will be easy. For instance what is at level 1 a product of two elements of a group $\mathbb{D}$ becomes at level 2, with a more detailed representation of those elements, a product of two 6×6 matrices of operators of a well defined form and, at level 3, may become a product of two 4×4 matrices with the well-known more detailed representation of displacements in coordinates (section 1.5.1) or another product of matrices when other representations are more convenient. As it will be explained in Chapter 1, this process of gradual translation integrates all the necessary differential calculus. The mathematical form of the relations holding at level 1 is preserved through all this process even if we should include a level for a computer language. It seems to be necessary to stress on the fact that, despite the rather abstract mathematical language used at level 1, it may be very readily translated into a programming language.

A frequent criticism against the improvements of the mathematical methods to model the mechanical systems by using more sophisticated mathematics says that the reward for the necessary efforts are out of proportion to the gain of their use. If such criticism would be fully justified their would not be so many attempts to get over the difficulty to solve kinematical problems or to build the dynamic equations for many body systems. And so many articles to come back to this problem presenting attempts by means of "new methods"[2]. This situation points out that a need for clarification arises. It is not easy to understand the mathematical structure which is behind the dynamic equations through their the very complicated expanded form in coordinates. The clarification will likely come from using an intrinsic formalism. The effectiveness of a direct approach by Lie group and Lie algebra theory is highlighted by the complete classification of the singularities of mechanisms; certainly, this classification would be extremely difficult to point out in the coordinate language. The same remark may be done about the investigations of the mathematical structure of the dynamic equation in the line of an easy interface between mathematical modelization and computer.

The organization of the book is the following:

Chapter 1 introduces the Lie group structure on the various examples of groups involved in mechanics of rigid body and rigid body systems from the standpoint of algebra and differential calculus. The mathematical techniques introduced in this

1 In practice but, in fact, at level 1 the form of the calculation is independent of the dimension.

2 The complexity of this problem, and consequently the need for truly new methods, was emphasized in Y. Papegay's thesis [PAP 92] where expanded forms of the dynamic equations which should be almost impossible to derive "by hand" are demonstrated.

chapter contain a complete system of rules for expanding all the calculations in kinematics and dynamics of articulated multibody systems. Those techniques are nothing but those of general theory of Lie groups applied to the Euclidean groups.

Chapter 2 presents the theory of dual numbers and dual vectors in an intrinsic form, showing that it is the study of a module structure on the Lie algebra of the Euclidean displacement group.

Chapter 3 completes the preceding Chapter 2 with some remarks on the so called transference principle.

In Chapter 4 the book starts with mechanics proper and points out the links between kinematics and the differential calculus in Lie groups in the typical case of a rigid bodies and chains of linked rigid body. This chapter also points out the relations between the standpoint of Lie groups and the more familiar expositions of mechanics relying on the models of rigid bodies as aggregates of particles.

Chapters 5 and 6 deal with kinematics of open and closed chains (*i.e.* mechanisms) and their singularities with examples of calculations based on the mathematical framework of this book.

Chapter 7 is a detailed presentation of the dynamics of the rigid body in the frame of Lie groups and of its links with the classical presentation of this matter. The fundamental law of dynamics is presented in Galilean and non-Galilean frames directly for a realistic body (not reduced to a massive particle). Everything indicates that the full system of the dynamic equations of a rigid body – in translation and in rotation and reduced to one equation [7.34] in dimension 6 – takes a very simple mathematical form easy to handle in the framework of multibody systems.

Chapter 8 deals with dynamics of rigid body systems. In particular as an example, a complete presentation of techniques to generate the dynamic equations of a tree-structured system.

Some exercises are proposed in order that the reader who will deal with them will become more familiar with the mathematical framework used in the book. In particular some proofs of theorems or propositions are left to the reader as exercises. An asterisk indicates an exercise or a question requiring the knowledge of rather technical tools in mathematics.

Each of these chapters includes an introduction with bibliographical references. We only mention here some general points. The systematic use of the (algebraic) structure of group in mechanisms theory, widely improved in the direction of the design of robots, was introduced by J. M. Hervé [HER 78] (1978) and in [HER 82], [HER 94]. The differential calculus on Lie groups with applications to kinematics was developped by A. Karger and J. Novak [KAR 85] (1985).

Kinematics and dynamics of multibody systems were presented in Lerbet [LER 88] (1988), Chevallier [CHE 86] (1984). Applications to dynamics of concrete complidcated mechanical systems appeared in C. Monnet and D. Chevallier [MON 84] (1984) or J.P. Mizzi [MIZ 92, MIZ 88] (1988). More recently Andreas Muller developped applications to mechanisms, with numerical algorithms (see for instance [MÜL 03, MÜL 14a, MÜL 14b]), Frederic Boyer and A. Shaukat (see [BOY 11] and [BOY 12]) developed applications to robotics fixed and mobile multibody systems including elements of Lie group theory in numerical methods.

Contents