GAME THEORY

Second Edition

Leon A. Petrosyan
Nikolay A. Zenkevich

World Scientific

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Preface

Game theory is a branch of modern applied mathematics that aims to analyze various problems of conflict between parties that have opposed, similar or simply different interests. A theory of games, introduced in 1921 by Émile Borel, was established in 1928 by John von Neumann and Oskar Morgenstern, to develop it as a means of decision making in complex economic systems. In their book "The Theory of Games and Economic Behaviour", published in 1944, they asserted that the classical mathematics developed for applications in mechanics and physics fail to describe the real processes in economics and social life. They have also seen many common factors such as conflicting interests, various preferences of decision makers, the dependence of the outcome for each individual from the decisions made by other individuals both in actual games and economic situations. Therefore, they named this new kind of mathematics game theory.

Games are grouped into several classes according to some important features. In our book we consider zero-sum two-person games, strategic n-person games in normal form, cooperative games, games in extensive form with complete and incomplete information, differential pursuit games and differential cooperative and non-cooperative n-person games.

There is no single game theory which could address such a wide range of "games". At the same time there are common optimality principles applicable to all classes of games under consideration, but the methods of effective computation of solutions are very different. It is also impossible to cover in one book all known optimality principles and solution concepts. For instance only the set of different "refinements" of Nash equilibria generates more than 20 new optimality principles. In this book we try to explain the principles which from our point of view are basic in game theory, and bring the reader to the ability to solve problems in this field of mathematics. We have included results published before in Petrosyan (1965), (1968), (1970), (1972), (1977), (1992), (1993); Petrosyan and Zenkevich (1986); Zenkevich and Marchenko (1987), (1990); Zenkevich and Voznyuk (1994); Kozlovskaya and Zenkevich (2010); Gladkova, Sorokina and Zenkevich (2013); Gao, Petrosyan and Sedakov (2014); Zenkevich and Zyatchin (2014); Petrosyan and Zenkevich (2015); Yeung and Petrosyan (2006), (2012); Petrosyan and Sedakov (2014); Petrosyan and Zaccour (2003); Zenkevich, Petrosyan and Yeung (2009).

The book is the second revised edition of Petrosyan and Zenkevich (1996).

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Chapter 1

Matrix Games

1.1 Definition of a Two-Person Zero-Sum Game in Normal Form

1.1.1 Definition. The system

$$\Gamma = (X, Y, K), \tag{1.1.1}$$

where X and Y are nonempty sets, and the function $K : X \times Y \to R^1$, is called a two-person zero-sum game in normal form.

The elements $x \in X$ and $y \in Y$ are called the *strategies* of Players 1 and 2, respectively, in the game Γ, the elements of the Cartesian product $X \times Y$ (i.e. the pairs of strategies (x, y), where $x \in X$ and $y \in Y$) are called *situations*, and the function K is the payoff of Player 1. Player 2's payoff in situation (x, y) is equal to $[-K(x, y)]$; therefore the function K is also called the *payoff function* of the game Γ and the game Γ is called a *zero-sum game*. Thus, in order to specify the game Γ, it is necessary to define the sets of strategies X, Y for Players 1 and 2, and the payoff function K given on the set of all situations $X \times Y$.

The game Γ is interpreted as follows. Players simultaneously and independently choose strategies $x \in X, y \in Y$. Thereafter Player 1 receives the payoff equal to $K(x, y)$ and Player 2 receives the payoff equal to $(-K(x, y))$.

Definition. The game $\Gamma' = (X', Y', K')$ is called a subgame of the game $\Gamma = (X, Y, K)$ if $X' \subset X, Y' \subset Y$, and the function $K' : X' \times Y' \to R^1$ is a restriction of function K on $X' \times Y'$.

This chapter focuses on two-person zero-sum games in which the strategy sets of the players' are finite.

1.1.2. Definition. Two-person zero-sum games in which both players have finite sets of strategies are called matrix games.

Suppose that Player 1 in matrix game (1.1.1) has a total of m strategies. Let us order the strategy set X of the first player, i.e. set up a one-to-one correspondence between the sets $M = \{1, 2, \dots, m\}$ and X. Similarly, if Player 2 has n strategies, it is possible to set up a one-to-one correspondence between the sets $N = \{1, 2, \dots, n\}$ and Y. The game Γ is then fully defined by specifying the matrix $A = \{a_{ij}\}$, where $a_{ij} = K(x_i, y_i)$, $(i, j) \in M \times N$, $(x_i, y_j) \in X \times Y, i \in M, j \in N$ (whence comes the name of the game — the matrix game). In this case the game Γ is realized as follows. Player 1 chooses row $i \in M$ and Player 2 (simultaneously and independently from Player 1) chooses column $j \in N$. Thereafter Player 1 receives the payoff (a_{ij}) and Player 2 receives the payoff $(-a_{ij})$. If the payoff is equal to a negative number, then we are dealing with the actual loss of Player 1.

Denote the game Γ with the payoff matrix A by Γ_A and call it the $(m \times n)$ game according to the dimension of matrix A. We shall drop index A if the discussion makes it clear what matrix is used in the game.

Strategies in the matrix game can be enumerated in different ways; therefore to each order relation, strictly speaking, corresponds its matrix. Accordingly, a finite two-person zero-sum game can be described by distinct matrices different from one another only by the order of rows and columns.

1.1.3. *Example 1* (Dresher, 1961). This example is known in literature as Colonel Blotto game. Colonel Blotto has m regiments and his enemy has n regiments. The enemy is defending two posts. The post will be taken by Colonel Blotto if when attacking the post he is more powerful in strength on this post. The opposing parties are two separate regiments between the two posts.

Define the payoff to the Colonel Blotto (Player 1) at each post. If Blotto has more regiments than the enemy at the post (Player 2), then his payoff at this post is equal to the number of the enemy's regiments plus one (the occupation of the post is equivalent to capturing of one regiment). If Player 2 has more regiments than Player 1 at the post, Player 1 loses his regiments at the post plus one (for the lost of the post). If each side has the same number of regiments at the post, it is a draw and each side gets zero. The total payoff to Player 1 is the sum of the payoffs at the two posts.

The game is zero-sum. We shall describe strategies of the players. Suppose that $m > n$. Player 1 has the following strategies: $x_0 = (m, 0)$ — to place all of the regiments at the first post; $x_1 = (m - 1, 1)$ — to place $(m-1)$ regiments at the first post and one at the second; $x_2 = (m-2, 2), \dots, x_{m-1} = (1, m-1), x_m = (0, m)$. The enemy (Player 2) has the following strategies: $y_0 = (n, 0)$, $y_1 = (n - 1, 1), \dots, y_n = (0, n)$.

Suppose that the Player 1 chooses strategy x_0 and Player 2 chooses strategy y_0. Compute the payoff a_{00} of Player 1 in this situation. Since $m > n$, Player 1 wins at the first post. His payoff is $n+1$ (one for holding the post). At the second post it is draw. Therefore, $a_{00} = n + 1$. Compute a_{01}. Since $m > n - 1$, then in the first post Player 1's payoff is $n - 1 + 1 = n$. Player 2 wins at the second post. Therefore, the loss of Player 1 at this post is one. Thus, $a_{01} = n - 1$. Similarly, we obtain $a_{0j} = n - j + 1 - 1 = n - j$, $1 \le j \le n$. Further, if $m - 1 > n$ then $a_{10} = n + 1 + 1 = n + 2$, $a_{11} = n - 1 + 1 = n$, $a_{1j} = n - j + 1 - 1 - 1 = n - j - 1$, $2 \le j \le n$. In a general case (for any m and n) the elements a_{ij}, $i = \overline{0, m}$, $j = \overline{0, n}$, of the payoff matrix are computed as follows:

$$a_{ij} = K(x_i, y_j) = \begin{cases} n+2 & \text{if } m-i > n-j, \quad i > j, \\ n-j+1 & \text{if } m-i > n-j, \quad i = j, \\ n-j-i & \text{if } m-i > n-j, \quad i < j, \\ -m+i+j & \text{if } m-i < n-j, \quad i > j, \\ j+1 & \text{if } m-i = n-j, \quad i > j, \\ -m-2 & \text{if } m-i < n-j, \quad i < j, \\ -i-1 & \text{if } m-i = n-j, \quad i < j, \\ -m+i-1 & \text{if } m-i < n-j, \quad i = j, \\ 0 & \text{if } m-i = n-j, \quad i = j. \end{cases}$$

Thus, with $m = 4, n = 3$, considering all possible situations, we obtain the payoff matrix A of this game:

$$A = \begin{matrix} & y_0 & y_1 & y_2 & y_3 \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\!\!\begin{bmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}.$$

Example 2. Game of Evasion [Gale (1960)]. Players 1 and 2 choose integers i and j from the set $\{1, \ldots, n\}$. Player 1 wins the amount $|i-j|$. The game is zero-sum. The payoff matrix is square $(n \times n)$ matrix, where $a_{ij} = |i-j|$. For $n = 4$, the payoff matrix A has the form

$$A = \begin{matrix} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{matrix}.$$

Example 3. Discrete Duel Type Game [Gale (1960)]. Players approach one another by taking n steps. After each step a player may or may not fire a bullet, but during the game he may fire only once. The probability that the player will hit his opponent (if he shoots) on the kth step is assumed to be k/n ($k \le n$).

A strategy for Player 1 (2) consists in taking a decision on shooting at the ith (jth) step. Suppose that $i < j$ and Player 1 makes a decision to shoot at the ith step and Player 2 makes a decision to shoot at the jth step. The payoff a_{ij} to Player 1 is then determined by

$$a_{ij} = \frac{i}{n} - \left(1 - \frac{i}{n}\right)\frac{j}{n} = \frac{n(i-j)+ij}{n^2}.$$

Thus the payoff a_{ij} is the difference in the probabilities of hitting the opponent and failing to survive. In the case $i > j$, Player 2 is the first to fire and $a_{ij} = -a_{ji}$. If however, $i = j$, then we set $a_{ij} = 0$. Accordingly, if we set $n = 5$, the game matrix multiplied by 25 has the form

$$A = \begin{bmatrix} 0 & -3 & -7 & -11 & -15 \\ 3 & 0 & 1 & -2 & -5 \\ 7 & -1 & 0 & 7 & 5 \\ 11 & 2 & -7 & 0 & 15 \\ 15 & 5 & -5 & -15 & 0 \end{bmatrix}.$$

Example 4. Attack-Defense Game. Suppose that Player 1 wants to attack one of the targets $c_1, \ldots, c_n$ having positive values $\tau_1 > 0, \ldots, \tau_n > 0$. Player 2 defends one of these targets. We assume that if the undefended target c_i is attacked, it is necessarily destroyed (Player 1 wins τ_i) and the defended target is hit with probability $1 > \beta_i > 0$ (the target c_i withstands the attack with probability $1 - \beta_i > 0$), i.e. Player 1 wins (on the average) $\beta_i\tau_i$, $i = 1, 2, \ldots, n$.

The problem of choosing the target for attack (for Player 1) and the target for defense (for Player 2) reduces to the game with the payoff matrix

$$A = \begin{bmatrix} \beta_1\tau_1 & \tau_1 & \ldots & \tau_1 \\ \tau_2 & \beta_2\tau_2 & \ldots & \tau_2 \\ \ldots & \ldots & \ldots & \ldots \\ \tau_n & \tau_n & \ldots & \beta_n\tau_n \end{bmatrix}.$$

Example 5. Discrete Search Game. There are n cells. Player 2 hide an object in one of n cells and Player 1 wishes to find it. In examining the ith cell, Player 1 exerts $\tau_i > 0$ efforts, and the probability of finding the object in the ith cell (if it is concealed there) is $0 < \beta_i \leq 1$, $i = 1, 2, \ldots, n$. If the object is found, Player 1 receives the amount α. The players' strategies are the numbers of cells wherein the players respectively hide and search for the object. Player 1's payoff is equal to the difference in the expected receipts and the efforts made in searching for the object. Thus, the problem of hiding and searching for the object reduces to the game with the payoff matrix

$$A = \begin{bmatrix} \alpha\beta_1 - \tau_1 & -\tau_1 & -\tau_1 & \ldots & -\tau_1 \\ -\tau_2 & \alpha\beta_2 - \tau_2 & -\tau_2 & \ldots & -\tau_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\tau_n & -\tau_n & -\tau_n & \ldots & \alpha\beta_n - \tau_n \end{bmatrix}.$$

Example 6. Noisy Search. Suppose that Player 1 is searching for a mobile object (Player 2) for the purpose of detecting it. Player 2's objective is the opposite one (i.e. he seeks to avoid being detected). Player 1 can move with velocities $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and Player 2 with velocities $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 3$, respectively. The range of the detecting device used by Player 1, depending on the velocities of the players is determined by the matrix

$$D = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \end{array}.$$

Strategies of the players are the velocities, and Player 1's payoff in the situation (α_i, β_j) is assumed to be the search efficiency $a_{ij} = \alpha_i \delta_{ij}$, $i = \overline{1,3}, j = \overline{1,3}$, where δ_{ij} is an element of the matrix D.

Then the problem of selecting velocities in a noisy search can be represented by the game with matrix

$$A = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} 4 & 5 & 6 \\ 6 & 8 & 10 \\ 3 & 6 & 9 \end{bmatrix} \end{array}.$$

Example 7. The Battle of the Bismarck See. The conflict can be modeled as the following 2×2 matrix game

$$\begin{array}{c} \\ N \\ S \end{array} \begin{array}{c} \begin{array}{cc} N & S \end{array} \\ \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \end{array}.$$

The first player is US Admiral Kenney and the second Japanese Admiral Imamura. The conflict happens in the South Pacific in 1943. Imamura has to transport troops across the Bismarck See to New Guinea, and his opponent Kenney wants to bomb the transport. Imamura has two possible choices: a shorter Northern route (N, 2 days) or a longer Southern route (S, 3 days). Kenney must choose one of this routs (N or S) to send his planes to. If he chooses the wrong route he can call back the planes and send them to another route, but the number of bombing days is reduced by 1. We assume, that the number of bombing days represents the payoff to Kenney in a positive sense to Imamura in negative sense.

1.2 Maximin and Minimax Strategies

1.2.1. Consider a two-person zero-sum game $\Gamma = (X, Y, K)$. In this game each of the players seeks to maximize his payoff by choosing a proper strategy. But for Player 1 the payoff is determined by the function $K(x, y)$, and for Player 2 it is determined by $(-K(x, y))$, i.e. the players' objectives are directly opposite. Note that the payoff of Player 1 (2) (the payoff function) is determined on the set of situations $(x, y) \in X \times Y$. Each situation, and hence