NUMERICAL FUNCTIONAL ANALYSIS

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Preface

The present book arose out of a year-long course in numerical functional analysis given from time to time since 1970 at the University of Wisconsin. This volume introduces the basic techniques of functional analysis and applies them to linear problems. It also lays the foundations for volume 2 which will consider boundary value problems for elliptic equations, and nonlinear problems.

Particular features include:

(1) The concepts of functional analysis are developed systematically, with frequent pauses to introduce applications. Detailed indexes are provided for the reader who wishes to use the material in a different order.

(2) Solutions are given for all the problems except the last few, which are very specialized. This should help students who have taken basic courses in analysis but have little experience of providing proofs.

(3) A large number of counterexamples are given, to show that various results are not true if the hypotheses are weakened. This introduces the reader to an important aspect of mathematics which is often neglected — one must not only *prove* conjectures but also *disprove* them.

Parts of the text were used as course material by John Halton, who made several useful suggestions. Ennio Stacchetti read the first eight chapters, solved many of the problems, and made an invaluable contribution.

The camera-ready copy was typed by Marilyn Wolff, who showed not only great skill but also limitless patience.

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Colin Cryer
Cambridge

Contents

1
Introduction

In this book we are concerned with the applications of functional analysis to numerical analysis. It is assumed that the reader is familiar with calculus, matrix theory, and basic numerical methods. Some knowledge of Lebesgue integration would be helpful, but the essential facts are summarized in Chapter 4. No knowledge of functional analysis is presumed, and the various aspects of functional analysis are developed as required. However, the emphasis is on the applications, and we sometimes state and explain, but do not prove, theorems with lengthy or uninteresting proofs in order to leave time to show how the theorems can be applied.

Functional analysis is a subject which has developed during the twentieth century. There were at first a number of scattered, but important, papers, and the subject perhaps first gained the stature of a discipline with the publication in 1932 of the monograph by S. Banach, *Théorie des opérations linéaires*. There is no generally accepted definition of functional analysis. From our point of view we may define functional analysis as infinite-dimensional analysis: that is, functional analysis extends, so far as possible, the concepts of matrix theory and calculus for a finite number of dimensions to an infinite number of dimensions. For example, consider the *Fredholm integral equation of the second kind*,

$$x(s) + \int_0^1 k(s,t)x(t)dt = f(s), \quad 0 \le s \le 1 . \tag{1.1}$$

Formally, this equation is rather similar to the matrix equation

$$x + Kx = f , \tag{1.2}$$

where x and f are n-vectors, and K is an $n \times n$-matrix. Indeed, many numerical methods of approximating eqn (1.1) will lead to an equation of the form (1.2). The solution x of eqn (1.2) involves only the finite number of components x_i of x, but the solution $x(s)$ of eqn (1.1) involves the infinitely many values $x(s)$, $0 \le s \le 1$. However, much of the theory of eqn (1.2) can be extended to eqn (1.1). Indeed, beginning with the beautiful paper of Fredholm (1900), the analysis of integral equations has been the source of many ideas in functional analysis.

The first volume of this book covers the basic theory of linear spaces. The second volume will begin by considering weak topologies, which serve as an introduction to Sobolev spaces, and then turn to the theory of nonlinear problems. While the subjects covered are standard, the emphasis given to the various topics is different to that in most texts on functional analysis for two reasons:

(1) We judge a particular topic by its usefulness rather than its depth or beauty. For example, the principle of uniform boundedness is discussed at great length in Chapter 5 because of its many applications.

(2) A numerical analyst may be thought of as a mathematician with one hand tied behind his back, because while a mathematician can use either constructive or non-constructive methods, a numerical analyst must use constructive methods. Therefore, a constructive approach such as a contraction mapping is much more useful than a non-constructive approach such as the Schauder fixed point theorem because the constructive approach immediately leads to a numerical method.

Numerical functional analysis began with the publication in 1948 in the Russian journal *Uspekhi Matem. Nauk* of a long paper by L. V. Kantorovich entitled 'Functional analysis and applied mathematics'. In the introduction to this paper Kantorovich wrote:

> 'Explicitly, we want to show that the ideas and methods of functional analysis may be used for the construction and analysis of effective practical algorithms for the solution of mathematical problems with just the same success as has attended their use for the theoretical investigation of these problems.'

It took quite a long time before the ideas of Kantorovich became widely known. A helping factor was the publication in 1952 by the U.S. Bureau of Standards of a translation due to C. D. Benster and G. E. Forsythe, but even this translation was a rarity. Another early reference, which drew attention to the value of using functional analysis to analyse numerical methods for matrix problems, was the book of Faddeeva, the first chapter of which was translated into English in 1952.

The goals of numerical functional analysis are:

(1) To simplify and unify by treating whole classes of problems at once.

(2) To bring the power of the general results of functional analysis to bear upon the problems of numerical analysis.

Numerical functional analysis may be expected to play an important rôle in the following situations:

(1) When the mathematics of a problem depends heavily upon functional analysis. For example, functional analysis plays an important rôle in the modern theory of partial differential equations, and it is therefore to be expected (and is indeed the case) that the theory of numerical methods for partial differential equations will also depend heavily upon functional analysis. (See Section 5.7)

(2) When whole classes of numerical methods are being considered, as, for example, when one considers all convergent quadrature formulae. (Section 5.3)

(3) When proving the existence of a numerical method with certain properties (e.g. Example 7.4).

On the other hand, numerical functional analysis may be expected to play a less important rôle when a specific method for a specific problem is being considered, and numerical functional analysis has nothing to say about the practical implementation of a numerical method.

The goals of numerical functional analysis are best illustrated by a famous example, Newton's method. Newton's method for solving the equation in a single real variable

$$f(x) = 0 , \tag{1.3}$$

is given by

$$x_{n+1} = x_n - f(x_n)/\dot{f}(x_n) . \tag{1.4}$$

If f is twice continuously differentiable then it is possible to show that Newton's method is quadratically convergent, and to establish conditions which ensure its convergence. How can we generalize Newton's method to the case when there are two equations

$$\begin{aligned} f(x,y) &= 0 , \\ g(x,y) &= 0 , \end{aligned} \tag{1.5}$$

and two unknowns x and y? Well, we note that eqn (1.4) is equivalent to

$$f(x_n) + (x_{n+1}-x_n)\dot{f}(x_n) = 0 .$$

That is, we have expanded $f(x)$ in a Taylor series about x_n, and retained only the first two terms. Applying the same argument to eqns (1.5) we obtain

$$\begin{aligned} f(x_n,y_n) + (x_{n+1}-x_n) \frac{\partial f}{\partial x} (x_n,y_n) + (y_{n+1}-y_n) \frac{\partial f}{\partial y} (x_n,y_n) = 0 , \\ g(x_n,y_n) + (x_{n+1}-x_n) \frac{\partial g}{\partial x} (x_n,y_n) + (y_{n+1}-y_n) \frac{\partial g}{\partial y} (x_n,y_n) = 0 , \end{aligned} \tag{1.6}$$

which is Newton's method for eqns (1.5). In his 1948 paper previously mentioned, Kantorovich considered Newton's method from the viewpoint of functional analysis. He established quadratic convergence, as well as conditions which ensure convergence. Kantorovich's proof held not only for eqn (1.6) but also for any finite number of dimensions, and, indeed, for many infinite-dimensional problems. Hence, in this case the methods of numerical functional analysis were triumphantly vindicated.

At present, the methods of functional analysis have so permeated numerical analysis that an understanding of functional analysis is essential for an understanding of modern numerical analysis. However, two warnings should be sounded:

(i) Functional analysis seeks to generalize so that in casting a particular problem into functional analysis form some essential features may be lost. For example, if the kernel $k(s,t)$ in eqn (1.1) is non-negative then certain numerical methods for solving eqn (1.1) may exhibit desirable properties such as monotone convergence, but these properties will be 'invisible' in any functional analysis approach which does not make use of the non-negativity of $k(s,t)$.

(ii) Ideas are not always generated by logical processes. An engineer may have a 'feeling' for a problem which may lead him to a method of solution. A functional analyst may not be led to the method of solution thought of by the engineer because his mind is working along different paths. The reverse is of course also true.

In summary, numerical analysis is not just a branch of functional analysis but, rather, functional analysis is a powerful tool for use in numerical analysis.

A detailed example of the beneficial interaction between physical ideas, functional analysis, and classical numerical analysis, is given by Radon's integral equation which is considered at length in (Section 9.7, p. 332).

2
Topological vector spaces

In this chapter we discuss the concept of a topological vector space which, as the name suggests, is a space possessing a topology (open sets) and permitting linear operations (as for vectors). Functional analysis is concerned with the properties of, and mappings between, topological vector spaces.

2.1 *PRELIMINARY REMARKS*

In elementary real analysis one is concerned with one topological vector space namely the *real line* R^1,

$$R^1 = \{x : -\infty < x < +\infty\} .$$

R^1 is the prototypal topological vector space, and we develop the concept of a topological vector space by generalizing properties of R^1. By successively introducing more and more assumptions, we obtain a hierarchy of spaces, beginning with the lowly topological spaces and ending with the Hilbert spaces. As more assumptions are made about a space, more properties can be deduced but fewer concrete examples exist. If too many assumptions are made there will be only one concrete example, the real line. There is, therefore, a balance to be achieved: one wishes to make enough assumptions to be able to derive interesting properties, but not so many assumptions as to exclude interesting applications.

The examples in this chapter and the following chapter are intended to illustrate basic ideas, and often have no direct connection with numerical analysis. In Chapter 4 we list and discuss most of the spaces which are widely used in numerical analysis. Later chapters give applications.

2.2 *SET THEORETIC NOTATION*

The following notation from set theory will be used (Halmos [1960]). The symbol ϵ denotes membership in a set so that $x \epsilon A$ means that x is an *element* or *point* of the set or *collection* A , that x *belongs* to A , and that A *contains* x . Sets are often defined by enumeration: the set $A = \{a,b,c\}$ consists of the three elements a, b, and c. Sets whose elements are indexed will often be called *families*; thus we may speak of a family of sets $\{A_i\}$ for $i \in I$, where I is an *index set*. The *empty set*, the set which contains no elements, is denoted by $\emptyset$.

We will often be concerned with a given collection of elements, one or more sets containing these elements, and one or more collections of these sets; as an aid in comprehension we will when possible denote the original elements by lower case Roman letters, the sets containing these elements by upper case Roman letters, and the collections of sets by Greek letters; for example,

$$x \in C \in \tau .$$

If $P(x)$ is a proposition concerning the elements x of a set A then $\{x \in A: P(x)\}$ denotes the set of elements of A for which $P(x)$ is true. For example, the set $\{x \in R^1: x \geq 0\}$ is the set of non-negative real numbers. The statement 'if $x \in \emptyset$ then P' is true for every proposition P .

If A and B are sets and $\{A_i: i \in I\}$ is a family of sets, then the *union* and *intersection* of A and B are denoted by $A \cup B$ and $A \cap B$, respectively, while the union and intersection of the A_i are denoted by

$$\bigcup_{i \in I} A_i \quad \text{or} \quad \cup A_i , \quad \text{and} \quad \bigcap_{i \in I} A_i \quad \text{or} \quad \cap A_i ,$$

respectively.

If every element of a set B is an element of a set A we say that A *includes* B , or A *contains* B , or B is a *subset* of A , and we write $A \supset B$ or $B \subset A$; in particular, $A \supset A$ for every set A . By convention, the empty set is a subset of every set. If $A \supset B$ and $A \neq B$, B is a *proper subset* of A . If $A \neq B$, A and B are *distinct*.

$A \setminus B$ or $A - B$ denotes the elements of A which do not belong to B ; $A \setminus B$ is the *complement of B with respect* *to A*. If A is understood from the context, we may speak of the *complement of B* and denote it by B^c.

The set A is *finite* if A contains a finite number of elements. The set A is *denumerable* or *countable* if there exists a one-to-one correspondence between the elements of A and a subset of the positive integers. The real interval [0,1] is not denumerable. (Natanson [1954, p. 13])

The symbol □ is used to denote the end of a logical entity, such as the proof of a theorem.

If P and Q are statements such that whenever P is true then Q is true, then P *implies* Q and we write $P \Rightarrow Q$. If, furtheremore, $Q \Rightarrow P$ then P and Q are *equivalent* and we write $P \Leftrightarrow Q$; another way of saying this is 'P if and only if Q' or, in abbreviated form, 'P *iff* Q'.

The logical operators AND, OR, and NOT will be used.

Logical notation helps to clarify the chain of arguments in the proof of theorems. To prove that $P \Rightarrow R$ it is often convenient to prove first that $P \Rightarrow Q$ and then that $Q \Rightarrow R$. If, furthermore, $R \Rightarrow P$, then $P \Leftrightarrow R$. We occasionally prove that $P \Rightarrow Q$ by proving the equivalent statement

$$\text{NOT } Q \Rightarrow \text{NOT } P .$$

2.3 *TOPOLOGICAL SPACES*

The first property of R^1 that we generalize is the concept of open and closed sets, such as the *open interval in* R^1

$$(a,b) = \{x: a < x < b\} ,$$

and the *closed interval in* R^1

$$[a,b] = \{x: a \le x \le b\} .$$

DEFINITION 2.1. A set X is a *topological space* with *topology* τ if τ is a collection of subsets of X satisfying the following three axioms:

(1) The empty set ∅ and the whole set X belong to τ.

(2) If $G_\alpha \in \tau$ for $\alpha \in A$, where A is an index set, then $\bigcup_{\alpha \in A} G_\alpha \in \tau$.

(3) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2 \in \tau$. □

The elements of a topology τ are called *open sets*, and axioms (2) and (3) of Definition 2.1 may be stated as follows: the union of arbitrarily many open sets is open; the intersection of finitely many open sets is open.

Strictly speaking, a topological space should be described as a pair $\{X,\tau\}$, but it is usually clear from the context, or by convention, which topology τ is associated with X, and in such cases we speak of the topological space X.

Some examples of topological spaces are:

EXAMPLE 2.1. $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$, where $a,b,$ and c are three elements. □

EXAMPLE 2.2. $X = R^1$. $G \in \tau$ iff for every $x \in G$ there exists an interval $(a,b) \subset G$ such that $x \in (a,b)$. This is the usual topology on R^1, and this topology will be used unless explicitly stated otherwise. □

There may be more than one topology on a set. Let two topologies be introduced on a set X by means of τ_1 and τ_2. If $\tau_1 \supset \tau_2$, we say that τ_1 is *stronger* (or *finer*) than τ_2, and that τ_2 is *weaker* (or *coarser*) than τ_1.

EXAMPLE 2.3. $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b,c\}\}$,
$\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$,
$\tau_3 = \{\emptyset, X, \{b\}, \{a,c\}\}$.

The topology τ_2 is stronger than both τ_1 and τ_3. □

Many spaces have two special topologies, the 'weak' topology and the 'strong' topology, which play an important rôle in the theory of such spaces.

Given a set X with two topologies τ_1 and τ_2 it can happen that neither topology is stronger than the other: in Example 2.3, $\tau_1 \not\subset \tau_3$ and $\tau_3 \not\subset \tau_1$, so that neither τ_1 nor τ_3 is stronger.

If $H = X \setminus G$ where G is an open subset of the topological space X, we say that H is *closed*. A set can be both open and closed. In Example 2.1 every element of

τ is both open and closed. There is a connection between closed sets and limits of sequences of elements which is discussed in Chapter 3 (see Theorem 3.2).

In Example 2.2 the topology on R^1 was defined by means of special sets, namely sets of the form (a,b). We now generalize this approach.

DEFINITION 2.2. Let X be a topological space with topology τ. A set $N_x \subset X$ is a *neighbourhood* (in the topology τ) of a point $x \in X$ if there is a set $G \in \tau$ such that $x \in G \subset N_x$. A collection β_x of neighbourhoods of a point x is called a *base of neighbourhoods* (in the topology τ) *of* x if β_x is non-empty and if for every neighbourhood V_x of x there exists $N_x \in \beta_x$ such that $N_x \subset V_x$. A family $\beta = \{\beta_x : x \in X\}$ of bases of neighbourhoods is called a *base of neighbourhoods for the topology* τ. □

REMARK 2.1. The subscript x on N_x in Definition 2.2 is not necessary and may be dropped, but it is often a helpful reminder of the link with x. □

REMARK 2.2. It follows immediately from Definition 2.2 that a non-empty open set is a neighbourhood of each of the points that it contains. □

REMARK 2.3. There are several slightly different definitions of neighbourhoods and bases in the literature. Sometimes a neighbourhood of x is defined to be an open set containing x, and a base β for a topology τ is defined to be a subfamily of τ such that for each x and each neighbourhood U of x there exists $V \in \beta$ satisfying $x \in V \subset U$. □

There may be more than one base for a topology as is shown by:

EXAMPLE 2.4. For R^1 (with the usual topology) one base of neighbourhoods $\beta' = \{\beta'_x\}$ is given by

$$\beta'_x = \{[a,b) : a < x < b\} .$$