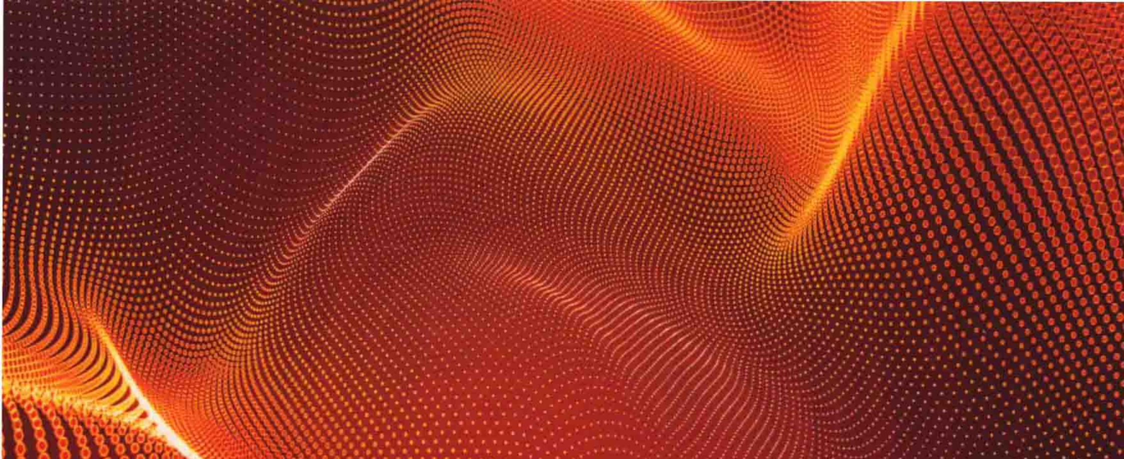

Nonlinear Theory of Elastic Plates

Anh Le van

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Mechanical Engineering and Solid Mechanics

This book presents three plate models, those of Cosserat, Reissner-Mindlin and Kirchhoff-Love respectively. From the point of view of the kinematics hypothesis, these range from the most general to the most restrictive; and from the point of view of the formulation of the model, these range from the simplest to the most complex.

While the explanation clearly demonstrates how the models are related to each other, it also allows the reader to approach each model independently.

This book is aimed at Mathematics and Engineering postgraduate students, researchers and mechanical and civil engineers. Furthermore, this book will also serve as a good introduction for those who wish to study shell models, as the tensor tools used are exactly the same and as the formulation of these theories is similar in all points to plate theory.

Anh Le van is Professor at the University of Nantes, France. His research at the GeM (Research Institute in Civil and Mechanical Engineering) includes membrane structures and, more specifically, the problems of contact and buckling of these structures.

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Series Editor
Noël Challamel

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First published 2017 in Great Britain and the United States by ISTE Press Ltd and Elsevier Ltd

ISTE Press Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

Elsevier Ltd
The Boulevard, Langford Lane
Kidlington, Oxford, OX5 1GB
UK

www.elsevier.com

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British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
Library of Congress Cataloging in Publication Data
A catalog record for this book is available from the Library of Congress
ISBN 978-1-78548-227-4

Nonlinear Theory of Elastic Plates

To my parents
To Nicole and Younnik
To Mai

Preface

This book is an introduction to nonlinear mechanics for plates. The non-linearities in play may be of geometric origin, due to finite deformations, or of material origin, arising from the hyperelastic nonlinear constitutive laws.

Why the nonlinear framework?

The nonlinear framework applies itself naturally to problems where plates undergo finite strains, finite displacements or finite rotations, for example the bending or forming of a metal sheet.

The nonlinear theory is also found to be necessary to account for the phenomenon of plate buckling (in statics) or of plate instability (in dynamics), even if the pre-critical strains and displacements are not significant. Indeed, we cannot restrict ourselves to a purely linear theory where equations are linearized too soon, but must instead carry out (at least partially) a non-linear analysis prior to linearizing the equations. This is a longer process but it is the only one that makes it possible to obtain the terms governing the buckling.

Synopsis of the book

1. In this book we will study three plate models. From the point of view of the kinematics hypothesis, these range from the most general to the most restrictive; from the point of view of the formulation of the model, these range from the simplest to the most complex:

   (a) The Cosserat plate model, whose kinematics is defined by the displacement field of the mid-surface and the field of the director vector, which is *a priori* arbitrary and independent of the displacement field of the mid-surface.

   (b) The Reissner-Mindlin plate model, where the director vector is constrained to be a unit vector.

   (c) The Kirchhoff-Love plate model, where the director vector must be both of unit length as well as orthogonal to the deformed mid-surface.

   While the explanation clearly demonstrates how the models are related to each other, it also allows the reader to approach each model independently, without referring to the other models.

   The governing equations of motion and the force boundary conditions will be obtained by means of the principle of virtual power. Interestingly, it can be seen that the results obtained at this stage are not subject to any hypothesis other than the kinematics assumption inherent to each plate model. Consequently, they are valid regardless of the amplitude of motion or the constitutive material. Results that depend on the material are presented separately.

2. The constitutive laws for plates will be established for hyperelastic materials. We will exclude more complicated behaviors such as elastoplasticity in finite deformations,

knowing that the difficulties related to these behaviors already exist in the 3D framework and are not specific to plate models.

Having obtained the constitutive laws, we will survey the whole set of equations and verify that there are as many equations as unknowns. This also is the time for us to review the different hypotheses adopted at different stages in the plate formulations.

3. We will finally study the linearization of the Kirchhoff-Love plate theory. The linear equations that result from this enable the study of the vibration of plates around a reference configuration that may be pre-stressed. Applied to the static case, the linearized equations also enable us to solve the problem of the buckling of plates.

It will be seen that several important subjects are not touched upon in this book. For example, we do not study the constitutive laws for orthotropic plates or stratified plates, finite elastoplastic deformations of plates, or plate finite elements, which are in themselves a large research domain. Nonetheless, the author hopes that this book may offer readers a solid foundation which will allow them to then venture further into the world of more complex nonlinear plate models. Furthermore, this book will also serve as a good introduction for those who wish to study shell models, as the tensor tools used are exactly the same and as the formulation of these theories is similar in all points to plate theory.

Finally, the author would like to thank Ms. Akhila Phadnis for her help with the English version of this book.

Anh Le van
Nantes, France
March 2017

Contents

1

Fundamentals of Tensor Theory

This chapter summarizes the definitions and results of the tensor operations that are used in plate theory. It can be divided into two parts:

1. Tensor algebra, where only algebraic operations such as addition and multiplication come into play.

2. Tensor analysis, which also involves the concept of derivatives.

The results are reviewed here without the proofs being worked out. For a detailed presentation, the readers are referred to mathematical works dedicated to tensor theory.

1.1. Tensor algebra

Let us consider a 3-dimensional Euclidean vector space E, endowed with the usual scalar product $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a}.\mathbf{b}$ and the Euclidian norm $||.||$. A basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$, not necessarily orthonormal, is chosen beforehand for E.

1.1.1. *Contravariant and covariant components of a vector*

Let $\mathbf{u}$ be a vector in E. The components of $\mathbf{u}$ in the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ are denoted by u^1, u^2, u^3 and we write $\boxed{\mathbf{u} = u^i \mathbf{g}_i}$, using the Einstein summation convention over any repeated index; here, the index i varies from 1 to 3. As the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is fixed, the vector $\mathbf{u}$ is determined by the coefficients u^1, u^2, u^3.

On the other hand, vector $\mathbf{u}$ is also determined by the three coefficients $\boxed{u_i \equiv \mathbf{u}.\mathbf{g}_i}$, $i \in \{1, 2, 3\}$. Indeed, we have

$$\forall i \in \{1, 2, 3\}, u_i \equiv \mathbf{u}.\mathbf{g}_i = (u^j \mathbf{g}_j).\mathbf{g}_i = u^j (\mathbf{g}_i.\mathbf{g}_j) \qquad [1.1]$$

By writing

$$\forall i, j \in \{1, 2, 3\}, \boxed{g_{ij} \equiv \mathbf{g}_i.\mathbf{g}_j} \qquad [1.2]$$

we can rewrite equation[1.1] in matrix form:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{Bmatrix} u^1 \\ u^2 \\ u^3 \end{Bmatrix} \qquad [1.3]$$

The 3×3 matrix $[g_{..}]$ with components $g_{ij}, i, j \in \{1, 2, 3\}$, is symmetrical. It is *invertible* because $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is a basis and, therefore, either of the triplets (u^1, u^2, u^3) or (u_1, u_2, u_3) allows us to determination of the other one.

Definitions. [1.4]

- *The contravariant components* of the vector $\mathbf{u}$ in the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ are the components u^1, u^2, u^3 in this basis. They are such that $\boxed{\mathbf{u} = u^i \mathbf{g}_i}$.
- *The covariant components* of the vector $\mathbf{u}$ in the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ are the coefficients u_1, u_2, u_3 defined by $\boxed{u_i \equiv \mathbf{u}.\mathbf{g}_i}$.

The notation convention with superscripts and subscripts (upper and lower indices) is systematically adopted in tensor theory. The advantage of this convention, as will be seen later on, is that it allows formulae to be easily read and systematically written.

Let us illustrate the concept of contravariant and covariant components in the two-dimensional space $\mathbb{R}^2$. We choose a basis $(\mathbf{g}_1, \mathbf{g}_2)$ for $\mathbb{R}^2$, formed of two *unit* vectors ($\|\mathbf{g}_1\| = \|\mathbf{g}_2\| = 1$), and we consider any vector $\mathbf{u}$. In Fig. 1.1:

- the contravariant components u^1, u^2 of vector $\mathbf{u}$ are the oblique components along $\mathbf{g}_1$ and $\mathbf{g}_2$.
- the covariant components u_1, u_2 are the orthogonal projection-value measures for $\mathbf{u}$ along $\mathbf{g}_1$ and $\mathbf{g}_2$.

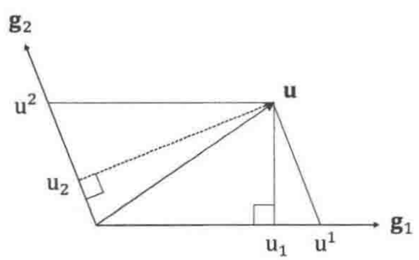

Figure 1.1: *Illustration in $\mathbb{R}^2$ of the contravariant and covariant components of a vector* $\mathbf{u}$

Using this example, we can see that the contravariant and covariant components are usually distinct. According to Eq. [1.3], the necessary and sufficient condition for them to be identical is that the matrix $[g_{..}]$ be equal to the identity matrix. That is, $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is *orthonormal*.

Theorem. Let $\mathbf{u}$ be a vector with contravariant components u^i and covariant components u_i; let $\mathbf{v}$ be a vector with contravariant components v^i and covariant components v_i. The scalar product of $\mathbf{u}$ and $\mathbf{v}$ is expressed by

$$\boxed{\mathbf{u}.\mathbf{v} = u^i v_i = u_i v^i} \qquad [1.5]$$

1.1.2. *Dual basis*

Notation. The components of the inverted matrix for $[g_{..}]$ are designated by g^{ij}:

$$\forall i,j \in \{1,2,3\}, \qquad \boxed{g^{ij} \equiv \left([g_{..}]^{-1}\right)_{ij} = g^{ji}} \qquad \Leftrightarrow \qquad \boxed{g^{ik} g_{kj} = \delta^i_j \quad \text{and} \quad g_{ik} g^{kj} = \delta^j_i}$$

where δ^i_j (also written as δ_{ij}) is Kronecker's symbol $\begin{cases} \delta^i_j = 1 & \text{if } i = j \\ \delta^i_j = 0 & \text{otherwise} \end{cases}$.

Theorem and definition. The family of vectors denoted by $\boxed{(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)}$ and defined by

$$\boxed{\mathbf{g}^i \equiv g^{ij}\mathbf{g}_j} \quad \Leftrightarrow \quad \boxed{\mathbf{g}_i \equiv g_{ij}\mathbf{g}^j} \qquad [1.6]$$

is *a basis* of E. It is also called the *dual basis* of $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$, as opposed to the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$, which is called the *primal basis*.

It must be pointed out that the dual basis is constructed *via* the following chain

$$\begin{array}{c}\text{primal basis}\\(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)\end{array} \rightarrow \text{matrix } [g_{\cdot\cdot}] \rightarrow \text{inverted matrix } [g^{\cdot\cdot}] \rightarrow \begin{array}{c}\text{dual basis}\\(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)\end{array}$$

The following theorem gives another characterization for the dual base in addition to definition [1.6].

Theorem.

- The vectors of the primal and dual bases are *orthogonal*:

$$\forall i, j \in \{1, 2, 3\}, \boxed{\mathbf{g}_i.\mathbf{g}^j = \delta_i^j}$$

- Conversely, any triplet of vectors $(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ which verifies $\mathbf{g}_i.\mathbf{a}^j = \delta_i^j$ is identical to the dual basis: $\forall i \in \{1, 2, 3\}, \mathbf{a}^i = \mathbf{g}^i$.

The following relationship is homologous to [1.2]:

Theorem.

$$\boxed{\mathbf{g}^i.\mathbf{g}^j = g^{ij}}$$

In general, the dual basis differs from the primal basis, except for the following special case:

Theorem. The primal basis is *orthonormal* $\Leftrightarrow$ the dual basis is *identical* to the primal basis.

1.1.3. *Different representations of a vector*

Theorem.

- The following relationships exist between the contravariant and covariant components of a vector $\mathbf{u}$:

$$\forall i \in \{1, 2, 3\}, \boxed{u_i = g_{ij}u^j}, \text{ conversely } \boxed{u^i = g^{ij}u_j}$$

  We thus lower or raise the indices using the matrices g_{ij} and g^{ij}.

- The following relationship is homologous to $u_i \equiv \mathbf{u}.\mathbf{g}_i$:

$$\forall i \in \{1, 2, 3\}, \boxed{u^i = \mathbf{u}.\mathbf{g}^i}$$

Theorem and definition. A vector can be expressed in either the primal basis or in the dual basis, as follows:

$$\boxed{\mathbf{u} \equiv u^i \mathbf{g}_i = u_i \mathbf{g}^i} \qquad [1.7]$$

These two forms are called *the contravariant and the covariant representations of* **u**.

From the previous theorem, we can also write $\mathbf{u} \equiv (\mathbf{u}.\mathbf{g}^i)\mathbf{g}_i = (\mathbf{u}.\mathbf{g}_i)\mathbf{g}^i$.

Theorem. The scalar product between vectors **u** and **v** may be written in different forms

$$\boxed{\mathbf{u}.\mathbf{v} = u^i v_i = u_i v^i = g_{ij} u^j v^i = g^{ij} u_j v_i} \qquad [1.8]$$

1.1.4. *Results related to the orientation of the 3D space*

The earlier results, written in three-dimensional space, may be generalized in the case of a space with n-dimensions (n being finite), using obvious notation changes. On the contrary, the results discussed in this section are only applicable to a 3-dimensional space.

As the space E is 3-dimensional, we can orient it and define a vector product (cross product) in it. We then obtain the following results related to a vector or mixed product.

Theorem.

$$\boxed{\mathbf{g}_1 \times \mathbf{g}_2 = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)\, \mathbf{g}^3} \quad \boxed{\mathbf{g}_2 \times \mathbf{g}_3 = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)\, \mathbf{g}^1} \quad \boxed{\mathbf{g}_3 \times \mathbf{g}_1 = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)\, \mathbf{g}^2}$$

Conversely

$$\boxed{\mathbf{g}^1 \times \mathbf{g}^2 = (\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)\, \mathbf{g}_3} \quad \boxed{\mathbf{g}^2 \times \mathbf{g}^3 = (\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)\, \mathbf{g}_1} \quad \boxed{\mathbf{g}^3 \times \mathbf{g}^1 = (\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)\, \mathbf{g}_2}$$

[1.9]

The vectors $\mathbf{g}_1, \mathbf{g}_2$ are *orthogonal* to vector $\mathbf{g}^3$, but they are not, in general, orthogonal to vector $\mathbf{g}_3$, Fig. 1.2.

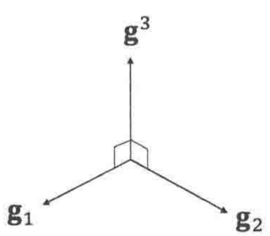

Figure 1.2: *Vector product of two vectors of the primal basis*

Theorem.

$$\boxed{(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3) = \frac{1}{(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)}} \qquad [1.10]$$

Therefore, the primal and dual bases have the *same* orientation.