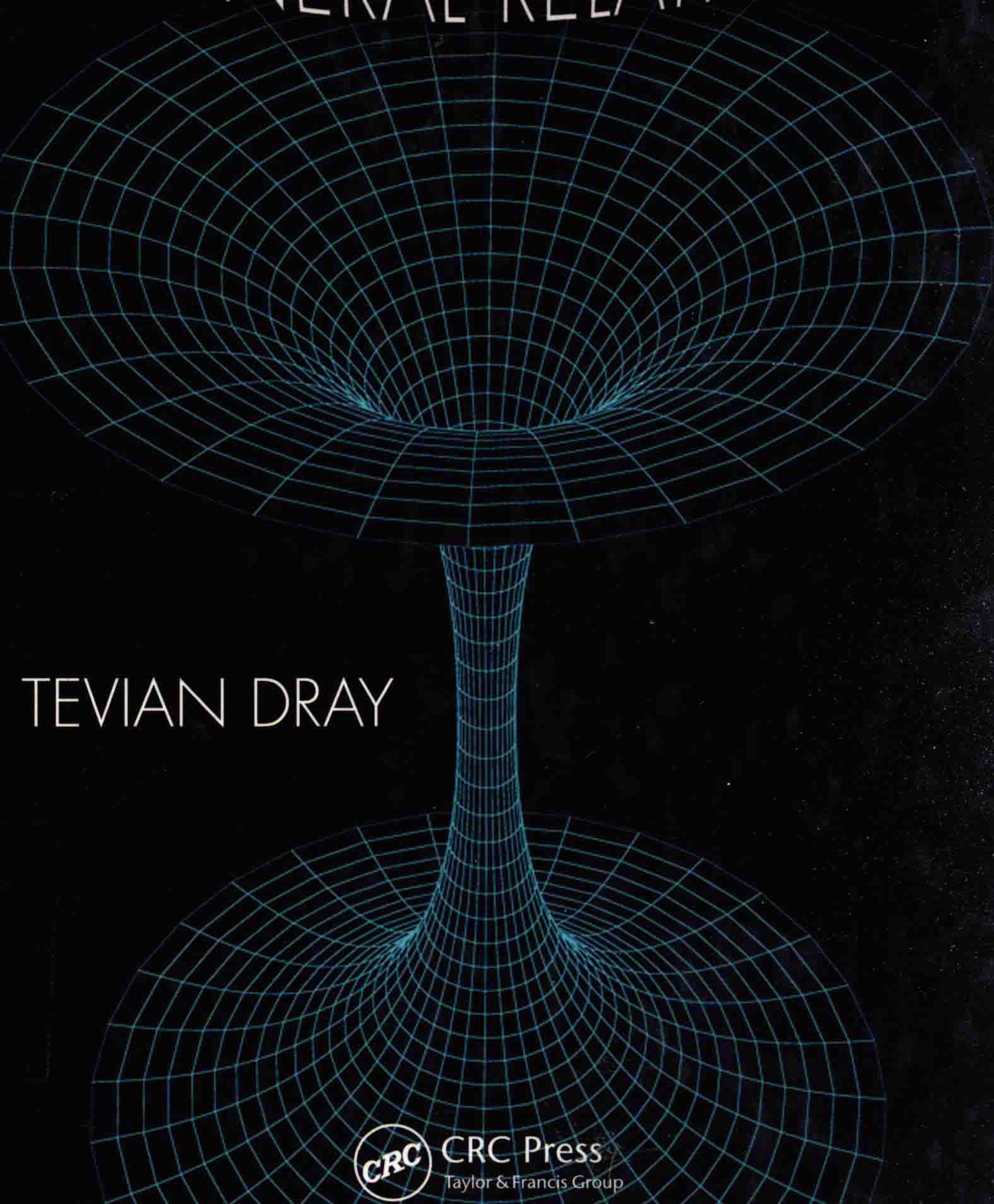
DIFFERENTIAL FORMS AND THE GEOMETRY OF GENERAL RELATIVITY



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Preface

This book contains two intertwined but distinct halves, each of which can in principle be read separately. The first half provides an introduction to general relativity, intended for advanced undergraduates or beginning graduate students in either mathematics or physics. The goal is to describe some of the surprising implications of relativity without introducing more formalism than necessary. "Necessary" is of course in the eye of the beholder, and this book takes a nonstandard path, using differential forms rather than tensor calculus, and trying to minimize the use of "index gymnastics" as much as possible.[1] This half of the book is itself divided into two parts, the first of which discusses the geometry of black holes, using little more than basic calculus. The second part, covering Einstein's equation and cosmological models, begins with an informal crash course in the use of differential forms, and relegates several messy computations to an appendix.

The second half of the book (Part III) takes a more detailed look at the mathematics of differential forms. Yes, it provides the theory behind the mathematics used in the first half of the book, but it does so by emphasizing conceptual understanding rather than formal proofs. The goal of this half of the book is to provide a language to describe curvature, the key geometric idea in general relativity.

[1]For the expert, the only rank-2 tensor objects that appear in the book are the metric tensor, the energy-momentum tensor, and the Einstein tensor, all of which are instead described as vector-valued 1-forms; the Ricci tensor is only mentioned to permit comparison with more traditional approaches.

PARTS I AND II: GENERAL RELATIVITY

As with most of my colleagues in relativity, I learned the necessary differential geometry the way mathematicians teach it, in a coordinate basis. It was not until years later, when trying to solve two challenging problems (determining when two given metrics are equivalent, and studying changes of signature) that I became convinced of the advantages of working in an orthonormal basis. This epiphany has since influenced my teaching at all levels, from vector calculus to differential geometry to relativity. The use of orthonormal bases is routine in physics, and was at one time the standard approach to the study of surfaces in three dimensions. Yet no modern text on general relativity makes fundamental use of orthonormal bases; at best, they calculate in a coordinate basis, then reinterpret the results using a more physical, orthonormal basis.

This book attempts to fill that gap.

The standard basis vectors used by mathematicians in vector analysis possess several useful properties: They point in the direction in which the (standard, rectangular) coordinates increase, they are orthonormal, and they are the same at every point. No other basis has all of these properties; whether working in curvilinear coordinates in ordinary, Euclidean geometry, or on the curved, Lorentzian manifolds of general relativity, some of these properties must be sacrificed.

The traditional approach to differential geometry, and as a consequence to general relativity, is to abandon orthonormality. In this approach, one uses a *coordinate basis*, in which, say, the basis vector in the θ direction corresponds to the differential operator that takes θ-derivatives. In other words, one defines the basis vector $\vec{\boldsymbol{e}}_\theta$ by an equation of the form

$$\vec{\boldsymbol{e}}_\theta \cdot \vec{\boldsymbol{\nabla}} f = \frac{\partial f}{\partial \theta}.$$

Physics, however, is concerned with measurement, and the physically relevant components of vector (and tensor) quantities are those with respect to an *orthonormal basis*. The fact that angular velocity is singular along the axis of symmetry is a statement about the use of angles to measure "distance", rather than an indication of a physical singularity. In relativity, where we don't always have a reliable intuition to fall back on, this distinction is especially important. We therefore work almost exclusively with orthonormal bases. Physics students will find our use of normalized vector fields such as

$$\hat{\boldsymbol{\theta}} = \frac{\vec{\boldsymbol{e}}_\theta}{|\vec{\boldsymbol{e}}_\theta|} = \frac{1}{r}\,\vec{\boldsymbol{e}}_\theta$$

familiar; mathematics students probably won't.

In both approaches, however, one must abandon the constancy of the basis vectors. Understanding how the basis vectors change from point to point leads to the introduction of a *connection*, and ultimately to *curvature*. These topics are summarized informally in Section 6.1, with a detailed discussion deferred until Part III.

We also follow an "examples first" approach, beginning with an analysis of the Schwarzschild geometry based on geodesics and symmetry, and only later discuss Einstein's equation. This allows the reader an opportunity to master the geometric reasoning essential to relativity before being asked to follow the more sophisticated arguments leading to Einstein's equation. Along the way, we discuss the standard applications of general relativity, including black holes and cosmological models.

No prior knowledge of physics is assumed in this book, although the reader will benefit from familiarity with Newtonian mechanics and with special relativity. This book does however assume a willingness to work with differential forms, which in turn requires familiarity with vector calculus and linear algebra. For the reader in a hurry, the essentials of both special relativity and differential forms are reviewed in Chapters 1 and 6, respectively.

PART III: DIFFERENTIAL FORMS

I took my first course in differential geometry as a graduate student. I got an A, but I didn't learn much. Many of my colleagues, including several non-mathematicians with a desire to learn the subject, have reported similar experiences.

Why should this be the case? I believe there are two reasons. First, differential geometry—like calculus—tends to be taught as a branch of analysis, not geometry. Everything is a map between suitable spaces: Curves and surfaces are parametrized; manifolds are covered with coordinate charts; tensors act on vectors; and so on. This approach may be good mathematics, but it is not very enlightening for beginners. Second, too much attention is given to setting up a general formalism, the tensor calculus. Differential geometry has been jokingly described as the study of those objects which are invariant under changes in notation, but this description is a shockingly accurate summary of the frustrations numerous students experience when trying to master the material.

This part of the book represents my attempt to do something different. The goal is to learn just enough differential geometry to be able to learn the basics of general relativity. Furthermore, the book is aimed not only at graduate students, but also at advanced undergraduates, not only in mathematics, but also in physics.

These goals lead to several key choices. We work with differential forms, not tensors, which are mentioned only in passing. We work almost exclusively in an orthonormal basis, both because it simplifies computations and because it avoids mistaking coordinate singularities for physical ones. And we are quite casual about concepts such as coordinate charts, topological constraints, and differentiability. Instead, we simply assume that our various objects are sufficiently well-behaved to permit the desired operations. The details can, and in my opinion should, come later.

This framework nonetheless allows us to recover many standard, beautiful results in $\mathbb{R}^3$. We derive formulas for the Laplacian in orthogonal coordinates. We discuss—but do not prove—Stokes' Theorem. We derive both Gauss's *Theorema Egregium* about intrinsic curvature and the Gauss–Bonnet Theorem relating geometry to topology. But we also go well beyond $\mathbb{R}^3$. We discuss the Cartan structure equations and the existence of a unique Levi-Civita connection. And we are especially careful *not* to restrict ourselves to Euclidean signature, using Minkowski space as a key example.

Yes, there is still much formalism to master. Furthermore, this classical approach is no longer standard—and certainly not as an introduction to relativity. I hope to have presented a coherent path to relativity for the interested reader, with some interesting stops along the way.

WEBSITE

A companion website for the book is available at

http://physics.oregonstate.edu/coursewikis/DFGGR/bookinfo

ACKNOWLEDGMENTS

First and foremost, I thank my wife and colleague, Corinne Manogue, for discussions and encouragement over many years. Her struggles with the traditional language of differential geometry, combined with her insight into how undergraduate physics majors learn—or don't learn—vector calculus have had a major influence on my increased use of differential forms and orthonormal bases in the classroom.

The use of differential forms, and especially of orthonormal bases, as presented in this book, represents a radical change in my own thinking. The relativity community consists primarily of physicists, yet they mostly learned differential geometry as I did, from mathematicians, in a coordinate basis. This gap is reminiscent of the one between the vector calculus taught by mathematicians, exclusively in rectangular coordinates, and the vector calculus used by physicists, mostly in curvilinear coordinates, and most definitely using orthonormal bases.

I have had the pleasure of working with Corinne for more than a decade to try to bridge this latter gap between mathematics and physics. Our joint efforts to make $d\vec{r}$ the key concept in vector calculus also led to my redesigning my differential geometry and relativity courses around the same idea.

My debt to Corinne is beyond words. She opened my eyes to the narrowness of my own vision of vector calculus, and, as a result, of differential geometry. Like any convert, I have perhaps become an extremist, for which only I am to blame. But the original push came from Corinne, to whom I am forever grateful.

I thank my department for encouraging the development of an undergraduate mathematics course in general relativity, then supporting this course over many years. I am grateful for the support and interest of numerous students, and for their patience as I experimented with several textbooks, including my own.

I am also grateful for the extensive support provided by the National Science Foundation for our work in vector calculus. Although this book is not directly related to those projects, there is no question that it was greatly influenced by my NSF-supported work. The interested reader is encouraged to browse the project websites for the Paradigms in Physics Project (http://physics.oregonstate.edu/portfolioswiki) and the Vector Calculus Bridge Project (http://www.math.oregonstate.edu/bridge), as well as our online vector calculus text [1].

How to Read This Book

There are several paths through this book, with different levels of mathematical sophistication. The two halves, on general relativity (Parts I and II) and differential forms (Part III), can be read independently, and in either order. I regularly teach a 10-week course on differential forms using Part III, followed by a 10-week course on general relativity, using Parts I and II but skipping most of Appendix A (and all of Appendix B). However, there are always a few students who take only the second course, and who make do with the crash course in Section 6.1.

Basic

The geometry of the spacetimes discussed in this book can be understood as geometric models without knowing anything about Einstein's field equation. This path requires only elementary manipulations starting from the line element, together with a single symmetry principle, but does not require any further knowledge of differential forms.

With these basic tools, a detailed study of the Schwarzschild geometry is possible, including its black hole properties, as is the study of simple cosmological models. However, the fact that these solutions solve Einstein's equation must be taken on faith, and the relationship between curvature, gravity, and tidal forces omitted.

Read:

- Chapters 1–5;
- Section 6.3;
- Chapters 9 and 10.

STANDARD

This path represents the primary route through the first half of the book, covering all of the content, but leaving out some of the details. Familiarity with differential forms is assumed, up to the level of being able to compute connection and curvature forms, at least in principle. However, familiarity with (other) tensors is not necessary, provided the reader is willing to treat the metric and Killing's equation informally, as simple products of infinitesimals.

Some further advanced mathematical topics can be safely skipped on this path, such as the discussion of the divergence of the metric and Einstein tensors in Appendix A. The reader who chooses this path may also choose to omit some computational details, such as the calculations of curvature given in Appendix A; such computations can also easily be done using computer algebra systems.

Read:

- Chapters 1–10.

EXPERT

Advanced readers will want to work through most of the computations in Appendix A.

Read:

- Chapters 1–10;
- Appendix A.

A TASTE OF DIFFERENTIAL FORMS

This path represents a quick introduction to differential forms, without many details.

Read one or both, in either order:

- Chapters 11 and 12;
- Sections 6.1 and 6.2.

A Course in Differential Forms

This path represents a reasonable if nonstandard option for an undergraduate course in differential geometry. Reasonable, because it includes both Gauss's *Theorema Egregium* about intrinsic curvature and the Gauss–Bonnet Theorem relating geometry to topology. Nonstandard, because it does not spend as much time on curves and surfaces in $\mathbb{R}^3$ as is typical. Advantages to this path are a close relationship to the language of vector calculus, and an introduction to geometry in higher dimensions and with non-Euclidean signature.

Read:

- Chapters 11–20.

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