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Differential topology with a view to applications

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Pitman Publishing

LONDON · SAN FRANCISCO · MELBOURNE

AMS Subject Classifications: (main) 58A05, 34–02
(subsidiary) 26A57, 34C–, 34D30, 58F–, 70–34

PITMAN PUBLISHING LTD
Pitman House, 39 Parker Street, London WC2B 5PB, UK

PITMAN PUBLISHING CORPORATION
6 Davis Drive, Belmont, California 94002, USA

PITMAN PUBLISHING PTY LTD
Pitman House, 158 Bouverie Street, Carlton, Victoria 3053, Australia

PITMAN PUBLISHING
COPP CLARK PUBLISHING
517 Wellington Street West, Toronto M5V 1G1, Canada

SIR ISAAC PITMAN AND SONS LTD
Banda Street, PO Box 46038, Nairobi, Kenya

PITMAN PUBLISHING CO SA (PTY) LTD
Craighall Mews, Jan Smuts Avenue, Craighall Park,
Johannesburg 2001, South Africa

Library of Congress Cataloging in Publication Data

Chillingworth, David.
Differential topology with a view to applications.

(Research notes in mathematics; 9)
Includes bibliographical references.
1. Differential topology. I. Title. II. Series.
QA613.6.C48 514'.7 76–28202
ISBN 0–273–00283–X

ISBN 0 273 00283 X

Reproduced and printed by photolithography and bound in
Great Britain at The Pitman Press, Bath

5811:97

Preface

With the increasing use of the language and machinery of differential topology in practical applications, many research workers in the physical, biological and social sciences are eager to learn some of the background to the subject, but are frustrated by the lack of any self-contained treatment that neither is too pure in approach nor is written at too advanced a level. Few people have the time and energy to embark on a systematic study in a field other than their own, and so this book is written with the purpose of making differential topology accessible in one volume as a working tool for applied scientists. It could also serve as a guide for graduate students finding their way around the subject before plunging into a more thorough treatment of one aspect or another. The book should perhaps be read more in the spirit of a novel (with a rather diffuse ending) than as a text-book.

The particular aim is to study the *global qualitative behaviour of dynamical systems*, although there are numerous byways and diversions along the route. A dynamical system is some system (economic, physical, biological ...) which evolves with time. Given a starting point, the system moves within a universe of possible states according to known or hypothesized laws, often describable locally by a formula for the 'infinitesimal' evolution, namely a differential equation. The *global* theory is the theory of all possible evolutions from all possible initial states, together with the way these fit together and relate to each other. *Qualitative* theory is concerned with the existence of constant (equilibrium) behaviour, periodic or recurrent behaviour, and long-term behaviour, together with questions of local and overall stability of the system. Global qualitative techniques, mainly stemming from the work of Henri Poincaré (1854-1912), are important both because precise quantitative theoretical solutions may in general be unobtainable, and because in any case a qualitative model is the basis of a sound mental picture without which mechanical calculation is highly dangerous.

The natural universe of evolution for a dynamical system is often a *differentiable manifold*; the evolution itself is a *flow* on the manifold, and a differential equation for infinitesimal evolution becomes a *vector field* on the manifold. Chapters 1-3 of the book are concerned with defining and explaining these terms, while Chapter 4 goes into the qualitative theory of flows on manifolds, ending with some discussion of bifurcation theory. There is an Appendix on basic terminology and notation for set theory.

Inevitably there are many topics which should have been included or developed but which would have expanded the volume to twice its size. Differential forms are hardly mentioned, singularity theory is only touched upon, and the fascinating terrain of general bifurcation theory for differential equations, including the Centre Manifold Theorem (one of the few really *practical* applications of differential topology), is left largely unexplored. I hope the tantalized reader will be able to follow up these topics via the references given.

Formal prerequisites are kept to a minimum. The ideas from topology and linear algebra that are needed are mostly developed from first principles, so that the basic requirements are hardly more than a familiarity with derivatives and partial derivatives in elementary calculus - although these, too, are defined in the text. The exceptions to this are *complex numbers*, which are assumed to be well-known objects to mathematically-minded scientists, and *determinants* and *eigenvalues* of matrices which may be less well-known to some but are everyday equipment for others. My excuse for this logical inconsistency is lack of space and the need to draw a line somewhere: I felt that it was more important to discuss carefully some of the fundamental ideas about linear spaces upon which the rest of the structure is built than to go on to techniques familiar to many people and in any case quite accessible elsewhere. In the first three Chapters the complex numbers feature mainly in examples and illustrations but in their rôles as eigenvalues they become crucial to the main plot in Chapter 4.

As overall references for the qualitative theory of dynamical systems, I suggest the now historic survey article by Smale [125] and the subsequent very readable lecture notes of Markus [73]. The excellent books on differential equations by Arnol'd [11] and Hirsch and Smale [55] are both directed towards the qualitative theory of flows on manifolds. For background on differential topology a recent and attractive text is Guillemin and Pollack [48]: there is also a forthcoming book by Hirsch [52]. The fascinating article on applications to fluid mechanics and relativity by Marsden, Ebin, and Fisher [75] is highly recommended (see also the introduction to differential topology by Stamm in the same volume).

The present book grew from a series of lectures given to a mixed audience of pure and applied mathematicians, engineers, physicists and economists at Southampton University in 1973/74. It is through the encouragement of several of these colleagues that I have expanded the lecture notes into book form, and I am grateful to them and others for helpful comments and criticisms. I am particularly indebted to Peter Stefan of the University College of North Wales, Bangor who carefully read the original notes and offered many detailed suggestions for improvement.

Despite all this assistance, I claim the credit for errors. I would also like to thank Professor Umberto Mosco and Professor Nicolaas Kuiper for hospitality at the Istituto Matematico della Università di Roma and the I.H.E.S., Bures-sur-Yvette, respectively, during visits to which I wrote up much of the notes. I am grateful also to Pitman Publishing for their interest in the book and patience during its production, to my wife Ann for tolerating the side-effects, and especially to Cheryl Saint and Jenny Medley for spending many long hours producing such a perfect typescript. Finally, my special thanks go to Les Lander for taking upon himself the task of drawing all the figures in the book, and obtaining such professional results in a short space of time.

David Chillingworth
Southampton, August 1976.

Contents

1 Basic topological ideas

1.1 THE CONCEPT OF A FUNCTION

We usually visualize a real-valued function of a real variable in terms of its graph:

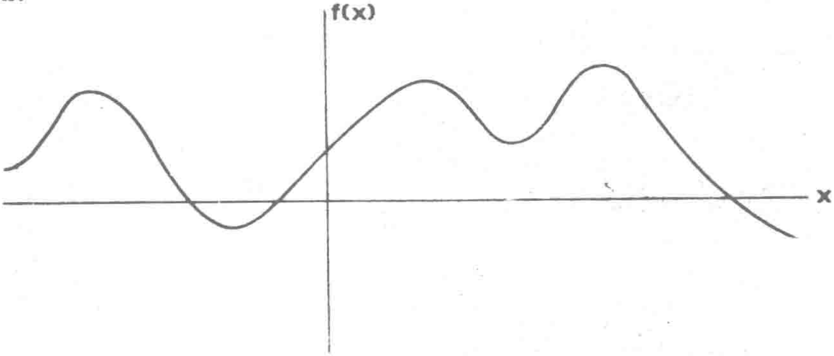

Figure 1

For a given number x (i.e. $x \in R$) the function f provides another real number $f(x)$. The graph consists of all points in a plane coordinatized by (x,y) which satisfy $y = f(x)$, or in formal notation

$$\text{graph}(f) = \{(x,y) \in R \times R \mid y = f(x)\} \quad .$$

Another 'picture' of the function, though less useful than the above, is the following:

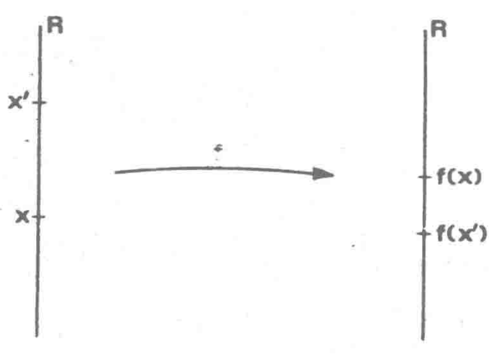

Figure 2

For a real function f of *two* real variables (x_1,x_2) the graph could be thought of as a landscape, with (x_1,x_2) as coordinates in a 'horizontal' plane and $f(x_1,x_2)$ being measured 'vertically'. Formally, we have $(x_1,x_2) \in R \times R = R^2$, $f(x_1,x_2) \in R$ and

$$\text{graph}(f) = \{((x_1,x_2),\ y) \in R^2 \times R = R^3 \mid y = f(x_1,x_2)\} .$$

Alternatively, we can picture f by a 'source and target' picture:

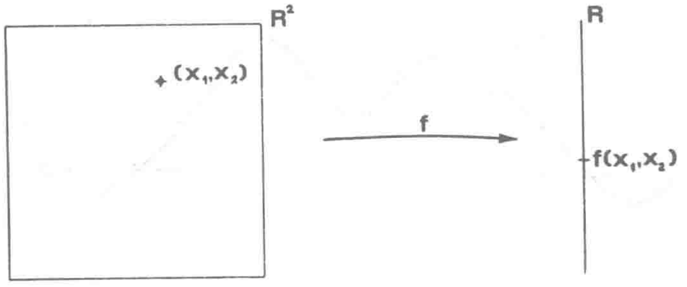

Figure 3

Now suppose we have two functions f_1, f_2 of two variables (x_1,x_2) . We can consider them both at the same time by writing

$$f(x_1,x_2) = (f_1(x_1,x_2),\ f_2(x_1,x_2)) \in R \times R = R^2 ,$$

so that f is a function from R^2 to R^2 .

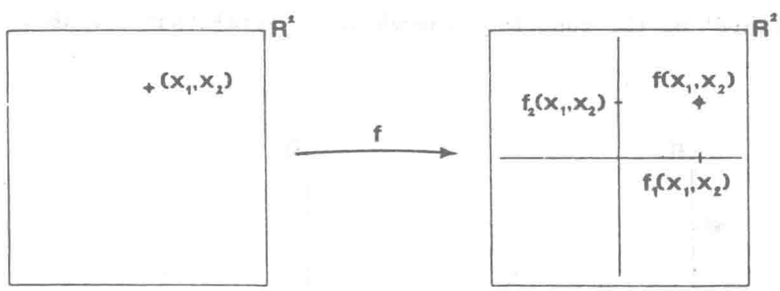

Figure 4

The 'graph' picture in this case is harder to visualize, since by analogy with the previous definition of graph we have

$$\text{graph}(f) = \{((x_1,x_2),\ (y_1,y_2)) \in R^2 \times R^2 \mid y_i = f_i(x_1,x_2)\ ,\ i = 1,\ 2\}$$

and so the graph is a subset of $R^2 \times R^2 = R^4$.

In a similar way, if we are given k functions of n variables we can put them all together to obtain a corresponding function from R^n to R^k . We write $x = (x_1, x_2, \ldots, x_n)$, let

$$f(x) = (f_1(x),\ f_2(x),\ \ldots,\ f_k(x)) \quad ,$$

and keep in mind the picture:

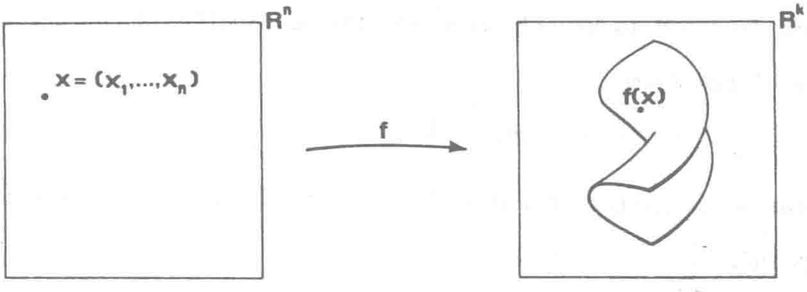

Figure 5

In formal notation this would be written as

$$f : R^n \to R^k \quad .$$

The graph of f would be a subset of $R^n \times R^k = R^{n+k}$, difficult to visualize and in general not yielding as much intuitive information about the behaviour of f as we might extract from Figure 5 by considering the way f causes R^n to be folded up and twisted inside R^k .
The techniques for analyzing such 'folds and twists' are those of topology and calculus together, or *differential topology*. Thus we see already how a study of differential topology may give useful insight into the ways in which k functions of n variables can mutually interact both in general

circumstances and in particular cases. Our aim in the first two chapters will be to develop some of these basic techniques.

Remarks

1. It is frequently necessary to consider functions which are not defined for all values of the variables $(x_1, x_2, \ldots, x_n)$ but only for $x = (x_1, x_2, \ldots, x_n)$ belonging to some subset U, say, of R^n.
In this case we of course write

$$f : U \to R^k \quad .$$

2. The word *function* is usually reserved for *real-valued functions* only, i.e. those of the form

$$f : (\text{something}) \to R \quad .$$

In other cases (e.g. for $f : U \to R^k$, $k > 1$) we tend to use the term *map* or *mapping*.

1.2 CONTINUITY

Roughly speaking, a map is continuous if by making a small perturbation in the input (i.e. the independent variables) you obtain only a small change in the output (i.e. the dependent variables). However, this is obviously far too vague a definition. For example, we would wish to think of the function $g : R \to R$ defined by $f(x) = 10^{23}x$ as being continuous (indeed, its graph is a straight line), but it could be argued that small changes in x produce very large changes in $g(x)$. The formal definition of continuity for functions $f : R \to R$ is as follows:

(a) *Continuity at a point* x_o . The function f is *continuous at* x_o if, given any positive number ε (thought of as admissible margin of error in the output), it is then possible to find another positive number δ such that perturbing x_o by less than δ causes $f(x_o)$ to vary by less than ε. Symbolically,

$$|x_o - x| < \delta \text{ implies } |f(x_o) - f(x)| < \varepsilon \ .$$

(b) *Continuity.* If we are considering f defined on some subset U of R, then we say that f is *continuous on* U if it is continuous at each point x_o belonging to U. If U is the whole of R or is in any case understood from the context then we simply say that f is *continuous*.

Note that the function g in the above example is continuous, since for any x_o and any ε we may take $\delta = 10^{23}\varepsilon$.

Remarks

1. It is tempting to try to combine (a) and (b) by saying (hopefully) that f is continuous on U if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon$$

for all x and y belonging to U. This is *not* the same as the previous definition, however. For example, the function $f : R \to R$ defined by $f(x) = x^2$ is continuous according to (a), (b) but it does not satisfy the 'hopeful' definition. The point is that for the genuine definition of continuity we must allow δ to depend on x_o (as well as on ε, of course). If we do not do this, but demand that the same δ should apply everywhere (given ε), then we have *uniform continuity* - a notion which is in fact of considerable importance in contexts involving approximating functions by other functions as for example in certain techniques of numerical analysis.

2. All 'standard' functions such as polynomials, exponentials, $\sin x$ and so on can easily be proved to be continuous where they are defined. The only hazard to continuity in combining them *ad libitum* is the risk of dividing by a function which vanishes somewhere.

It is easy to see how the definition of continuity will go over to functions $f : R^n \to R^k$, since all that is necessary is to replace the modulus by the euclidean distance from the origin in R^n or R^k (see Example 3 below). However, we shall need to study the 'small change in input gives small change in output' problem in situations where the input and output may be rather more complicated than numbers or n-tuples of real numbers; for example, they may be differential equations, or perhaps collections of functions. Therefore we want to generalize the definition of continuity so that it applies to maps $f : A \to B$ where A and B are sets other than R^n or subsets of R^n. To do this we need notions of *distance* in both sets A, B; then we could simply say

(a) $f : A \to B$ is *continuous at* $x_o \in A$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if the distance from x to x_o (in A) is less than δ then the distance from $f(x)$ to $f(x_o)$ (in B) is less than ε;

(b) $f : A \to B$ is *continuous* if it is continuous at every point in A.

Now it turns out that the minimal properties that a 'distance function' d on a set S needs to have in order to reflect adequately the basic relationships of distance in euclidean space are these:-

(1) $d(s,s')$ is always $\geqslant 0$, and $= 0$ when and only when $s = s'$;

(2) $d(s,s') = d(s',s)$ for all s and s' in S ;

(3) $d(s,s'') \leq d(s,s') + d(s',s'')$ for all s, s' and s'' in S .

(This last property is known as the *triangle inequality*.) A distance function satisfying (1), (2) and (3) is called a *metric*.

DEFINITION

A function d *which satisfies (1), (2) and (3) above is called a* metric *on* S. *A set* S *together with a particular metric on it is called a* metric space.

Note that d is actually a function from $S \times S$ to R, and not a function on S itself. The image of d is contained in the set R_o^+ of non-negative real numbers (by (1)), and so we could write $d : S \times S \to R_o^+$.

EXAMPLES of metric spaces

1. $S = R$, $d(x,y) = |x - y|$ (the prototype example).

2. $S = R^2$, $d(x,y) = \left|(x_1 - y_1)^2 + (x_2 - y_2)^2\right|^{\frac{1}{2}}$.

3. $S = R^n$, $d(x,y) = \left|\sum_{i=1}^{n} (x_i - y_i)^2\right|^{\frac{1}{2}}$.

Examples 1 and 2 are special cases of Example 3, known as the *euclidean* metric or *usual* metric in R^n .

4. $S = \{\text{bounded functions } f : [a,b] \to R\}$

$$d(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)| .$$

5. $S = \{\text{differentiable functions } f : (a,b) \to R \text{ with bounded derivative}\}$

$$d(f,g) = \sup_{a < x < b} |f(x) - g(x)| + \sup_{a < x < b} |f'(x) - g'(x)| .$$

Examples 4 and 5 are simple examples of *function spaces*, i.e. spaces whose elements are themselves functions or maps defined on other spaces. Example 6 is also a type of function space.

6. S = {systems of differential equations of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,x_2)\\ \dot{x}_2 &= f_2(x_1,x_2)\end{aligned} \qquad (F)$$

defined on R^2 and such that f_1, f_2 are bounded continuous functions}

$$d(F,G) = \sup_{(x_1,x_2)\,\varepsilon\,R^2} \left| (f_1(x_1,x_2) - g_1(x_1,x_2))^2 + (f_2(x_1,x_2) - g_2(x_1,x_2))^2 \right|^{\frac{1}{2}}$$

where F, G are defined by f's, g's respectively.

There are two important general types of example:

7. Let S be any set, and let T be a subset of S. Suppose that S is equipped with a metric d. Then T naturally inherits a metric from d, called the *induced* metric. Formally, if we write

$$d : S \times S \to R_o^+$$

then the induced metric is the restriction $d|T \times T$.

8. Let S be any set, and define d by

$$\left.\begin{aligned} d(x,x) &= 0 \quad \text{for all} \quad x \ \varepsilon\ S \\ d(x,y) &= 1 \quad \text{whenever} \quad x \neq y \end{aligned}\right\} .$$

It is easy to verify that this satisfies the rules (1), (2), (3) for a metric. It is known as the *discrete metric* on S. Note that we do not