

Graduate Texts in Mathematics

GTM

Loukas Grafakos

Classical Fourier Analysis

Third Edition

经典傅里叶分析

第3版

Springer



世界图书出版公司
www.wpcbj.com.cn

Loukas Grafakos

Classical Fourier Analysis

Third Edition



Springer

Loukas Grafakos
Department of Mathematics
University of Missouri
Columbia, MO, USA

ISSN 0072-5285

ISBN 978-1-4939-1193-6

DOI 10.1007/978-1-4939-1194-3

Springer New York Heidelberg Dordrecht London

ISSN 2197-5612 (electronic)

ISBN 978-1-4939-1194-3 (eBook)

Library of Congress Control Number: 2014946585

Mathematics Subject Classification (2010): 42Axx, 42Bxx

© Springer Science+Business Media New York 2000, 2008, 2014

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Reprint from English language edition:

Classical Fourier Analysis 3rd ed.

by Loukas Grafakos

Copyright © Springer Science+Business Media New York 2000, 2008, 2014

Springer is a part of Springer Science+Business Media

All Rights Reserved

This reprint has been authorized by Springer Science & Business Media for distribution in China Mainland only and not for export therefrom.

图书在版编目 (C I P) 数据

经典傅里叶分析 = Classical Fourier Analysis Third Edition : 第 3 版 :

英文 / (美) L. 格拉法克斯 (L. Grafakos) 著 . — 影印本 . —

北京 : 世界图书出版有限公司北京分公司 , 2017.4

ISBN 978-7-5192-2608-4

I . ①经… II . ①L… III . ①傅里叶分析—英文 IV . ①O174.2

中国版本图书馆 CIP 数据核字 (2017) 第 068708 号

中文书名 经典傅里叶分析 第 3 版
英文书名 Classical Fourier Analysis 3rd ed.

著 者 Loukas Grafakos
责任编辑 刘 慧 高 蓉
装帧设计 任志远

出版发行 世界图书出版有限公司北京分公司
地 址 北京市东城区朝内大街 137 号
邮 编 100010
电 话 010-64038355 (发行) 64037380 (客服) 64033507 (总编室)
网 址 <http://www.wpcbj.com.cn>
邮 箱 wpcbjst@vip.163.com
销 售 新华书店
印 刷 三河市国英印务有限公司
开 本 711 mm × 1245 mm 1/24
印 张 27.5
字 数 528 千字
版 次 2017 年 8 月第 1 版 2017 年 8 月第 1 次印刷
版权登记 01-2016-7821
国际书号 ISBN 978-7-5192-2608-4
定 价 99.00 元

版权所有 翻印必究

(如发现印装质量问题, 请与本公司联系调换)

Loukas Grafakos

Classical Fourier Analysis

Third Edition

Advisory Board

- John Adams, Williams College, Williamstown, MA, USA
 Alberto Azeiteiro, University of British Columbia, Vancouver, BC, Canada
 Paul Brumer, University of Toronto, Toronto, ON, Canada
 John M. Gamboa, The University of Texas at Austin, Austin, TX, USA
 Roger E. Howe, Yale University, New Haven, CT, USA
 David Jerison, Massachusetts Institute of Technology, Cambridge, MA, USA
 Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI, USA
 Jill Lagarias, University of Michigan, Ann Arbor, MI, USA
 Frits Sturtevant, University of Minnesota, Minneapolis, MN, USA
 Anne Wilkinson, University of Chicago, Chicago, IL, USA



Graduate Texts in Mathematics bridge the gap between research and teaching. They are written by leading experts in the field and are designed to be used as textbooks in graduate courses. The volumes are carefully written to teaching style and highlight the key results of the theory. Although these books are primarily used as textbooks in graduate courses, they are also suitable for individual study.

Graduate Texts in Mathematics

Series Editors:

Sheldon Axler

San Francisco State University, San Francisco, CA, USA

Kenneth Ribet

University of California, Berkeley, CA, USA

Advisory Board:

Colin Adams, *Williams College, Williamstown, MA, USA*

Alejandro Adem, *University of British Columbia, Vancouver, BC, Canada*

Ruth Charney, *Brandeis University, Waltham, MA, USA*

Irene M. Gamba, *The University of Texas at Austin, Austin, TX, USA*

Roger E. Howe, *Yale University, New Haven, CT, USA*

David Jerison, *Massachusetts Institute of Technology, Cambridge, MA, USA*

Jeffrey C. Lagarias, *University of Michigan, Ann Arbor, MI, USA*

Jill Pipher, *Brown University, Providence, RI, USA*

Fadil Santosa, *University of Minnesota, Minneapolis, MN, USA*

Amie Wilkinson, *University of Chicago, Chicago, IL, USA*

Graduate Texts in Mathematics bridge the gap between passive study and creative understanding, offering graduate-level introductions to advanced topics in mathematics. The volumes are carefully written as teaching aids and highlight characteristic features of the theory. Although these books are frequently used as textbooks in graduate courses, they are also suitable for individual study.

For further volumes:

<http://www.springer.com/series/136>

Preface

To Suzanne

The great response to the public of *Classical and Modern Fourier Analysis* in 2004 has been especially gratifying to me. I was delighted when Springer offered to publish the second edition in 2009 in two volumes: *Classical Fourier Analysis, 2nd Edition*, and *Modern Fourier Analysis, 2nd Edition*. I am now elated to have the opportunity to write the current third edition of these books, which Springer has also kindly offered to publish. The third edition was born from my desire to improve the exposition in several places, fix a few inaccuracies, and add some new material. I have been very fortunate to receive several hundred e-mail messages that helped me improve the proofs and locate mistakes and misprints in the previous editions.

In this edition, I maintain the same style as in the previous ones. The proofs contain details that unavoidably make the reading more cumbersome. Although it will behoove many readers to skim through the more technical aspects of the presentation and concentrate on the flow of ideas, the fact that details are present will be comforting to some. (This last sentence is based on my experience as a graduate student.) Readers familiar with the second edition will notice that the chapter on weights has been moved from the second volume to the first.

This first volume *Classical Fourier Analysis* is intended to serve as a text for a one-semester course with prerequisites of measure theory, Lebesgue integration, and complex variables. I am aware that this book contains significantly more material than can be taught in a semester course; however, I hope that this additional information will be useful to researchers. Based on my experience, the following list of sections (or parts of them) could be taught in a semester without affecting the logical coherence of the book: Sections 1.1, 1.2, 1.3, 2.1, 2.1.1, 2.3, 3.1, 3.2, 3.3, 4.4, 4.5, 5.1, 5.2, 5.3, 5.5, 5.6, 6.1, 6.2.

A long list of people have assisted me in the preparation of this book, but I remain solely responsible for any misprints, mistakes, and omissions contained therein. Please contact me directly (gralakos@missouri.edu) if you have corrections or comments. Any corrections to this edition will be posted to the website

<http://math.missouri.edu/~loukas/FourierAnalysis.html>.

Preface

The great response to the publication of my book *Classical and Modern Fourier Analysis* in 2004 has been especially gratifying to me. I was delighted when Springer offered to publish the second edition in 2008 in two volumes: *Classical Fourier Analysis, 2nd Edition*, and *Modern Fourier Analysis, 2nd Edition*. I am now elated to have the opportunity to write the present third edition of these books, which Springer has also kindly offered to publish. The third edition was born from my desire to improve the exposition in several places, fix a few inaccuracies, and add some new material. I have been very fortunate to receive several hundred e-mail messages that helped me improve the proofs and locate mistakes and misprints in the previous editions.

In this edition, I maintain the same style as in the previous ones. The proofs contain details that unavoidably make the reading more cumbersome. Although it will behoove many readers to skim through the more technical aspects of the presentation and concentrate on the flow of ideas, the fact that details are present will be comforting to some. (This last sentence is based on my experience as a graduate student.) Readers familiar with the second edition will notice that the chapter on weights has been moved from the second volume to the first.

This first volume *Classical Fourier Analysis* is intended to serve as a text for a one-semester course with prerequisites of measure theory, Lebesgue integration, and complex variables. I am aware that this book contains significantly more material than can be taught in a semester course; however, I hope that this additional information will be useful to researchers. Based on my experience, the following list of sections (or parts of them) could be taught in a semester without affecting the logical coherence of the book: Sections 1.1, 1.2, 1.3, 2.1, 2.2., 2.3, 3.1, 3.2, 3.3, 4.4, 4.5, 5.1, 5.2, 5.3, 5.5, 5.6, 6.1, 6.2.

A long list of people have assisted me in the preparation of this book, but I remain solely responsible for any misprints, mistakes, and omissions contained therein. Please contact me directly (grafakosl@missouri.edu) if you have corrections or comments. Any corrections to this edition will be posted to the website

<http://math.missouri.edu/~loukas/FourierAnalysis.html>

which I plan to update regularly. I have prepared solutions to all of the exercises for the present edition which will be available to instructors who teach a course out of this book.

Athens, Greece,
March 2014

Loukas Grafakos

Acknowledgments

I am extremely fortunate that several people have pointed out errors, misprints, and omissions in the previous editions of the books in this series. All these individuals have provided me with invaluable help that resulted in the improved exposition of the text. For these reasons, I would like to express my deep appreciation and sincere gratitude to the all of the following people.

First edition acknowledgements: Georgios Alexopoulos, Nakhlé Asmar, Bruno Calado, Carmen Chicone, David Cramer, Geoffrey Diestel, Jakub Duda, Brenda Frazier, Derrick Hart, Mark Hoffmann, Steven Hofmann, Helge Holden, Brian Hollenbeck, Petr Honzík, Alexander Iosevich, Tunde Jakab, Svante Janson, Ana Jiménez del Toro, Gregory Jones, Nigel Kalton, Emmanouil Katsoprinakis, Dennis Kletzing, Steven Krantz, Douglas Kurtz, George Lobell, Xiaochun Li, José María Martell, Antonios Melas, Keith Mersman, Stephen Montgomery-Smith, Andrea Nahmod, Nguyen Cong Phuc, Krzysztof Oleszkiewicz, Cristina Pereyra, Carlos Pérez, Daniel Redmond, Jorge Rivera-Noriega, Dmitriy Ryabogin, Christopher Sansing, Lynn Savino Wendel, Shih-Chi Shen, Roman Shvidkoy, Elias M. Stein, Atanas Stefanov, Terence Tao, Erin Terwilleger, Christoph Thiele, Rodolfo Torres, Deanie Tourville, Nikolaos Tzirakis, Don Vaught, Igor Verbitsky, Brett Wick, James Wright, and Linqiao Zhao.

Second edition acknowledgements: Marco Annoni, Pascal Auscher, Andrew Bailey, Dmitriy Bilyk, Marcin Bownik, Juan Caverio de Carondelet Fiscowich, Leonardo Colzani, Simon Cowell, Mita Das, Geoffrey Diestel, Yong Ding, Jacek Dziubanski, Frank Ganz, Frank McGuckin, Wei He, Petr Honzík, Heidi Hulsizer, Philippe Jaming, Svante Janson, Ana Jiménez del Toro, John Kahl, Cornelia Kaiser, Nigel Kalton, Kim Jin Myong, Doowon Koh, Elena Koutcherik, David Kramer, Enrico Laeng, Sungyun Lee, Qifan Li, Chin-Cheng Lin, Liguang Liu, Stig-Olof Londen, Diego Maldonado, José María Martell, Mieczysław Mastyło, Parasar Mohanty, Carlo Morpurgo, Andrew Morris, Mihail Mourgoglou, Virginia Naibo, Tadahiro Oh, Marco Peloso, Maria Cristina Pereyra, Carlos Pérez, Humberto Rafeiro, Maria Carmen Reguera Rodríguez, Alexander Samborskiy, Andreas Seeger, Steven Senger, Sumi Seo, Christopher Shane, Shu Shen, Yoshihiro Sawano, Mark Spencer, Vladimir Stepanov, Erin Terwilleger, Rodolfo H. Torres, Suzanne Tourville,

Ignacio Uriarte-Tuero, Kunyang Wang, Huoxiong Wu, Kôzô Yabuta, Takashi Yamamoto, and Dachun Yang.

Third edition acknowledgments: Marco Annoni, Mark Ashbaugh, Daniel Azagra, Andrew Bailey, Árpád Bényi, Dmitriy Bilyk, Nicholas Boros, Almut Burchard, María Carro, Jameson Cahill, Juan Caverio de Carondelet Fiscowich, Xuemei Chen, Andrea Fraser, Shai Dekel, Fausto Di Biase, Zeev Ditzian, Jianfeng Dong, Oliver Dragičević, Sivaji Ganesh, Friedrich Gesztesy, Zhenyu Guo, Piotr Hajłasz, Danqing He, Andreas Heinecke, Steven Hofmann, Takahisa Inui, Junxiong Jia, Kasinathan Kamalavasanthi, Hans Koelsch, Richard Laugesen, Kaitlin Leach, Andrei Lerner, Yiyu Liang, Calvin Lin, Liguang Liu, Elizabeth Loew, Chao Lu, Richard Lynch, Diego Maldonado, Lech Maligranda, Richard Marcum, Mieczysław Mastyło, Mariusz Mirek, Carlo Morpurgo, Virginia Naibo, Hanh Van Nguyen, Seungly Oh, Tadahiro Oh, Yusuke Oi, Lucas da Silva Oliveira, Kevin O'Neil, Hesam Oveys, Manos Papadakis, Marco Peloso, Carlos Pérez, Jesse Peterson, Dmitry Prokhorov, Amina Ravi, Maria Carmen Reguera Rodríguez, Yoshihiro Sawano, Mirye Shin, Javier Soria, Patrick Spencer, Marc Strauss, Krystal Taylor, Naohito Tomita, Suzanne Tourville, Rodolfo H. Torres, Fujioka Tsubasa, Ignacio Uriarte-Tuero, Brian Tuomanen, Shibi Vasudevan, Michael Wilson, Dachun Yang, Kai Yang, Yandan Zhang, Fayou Zhao, and Lifeng Zhao.

Among all these people, I would like to give special thanks to an individual who has studied extensively the two books in the series and has helped me more than anyone else in the preparation of the third edition: Danqing He. I am indebted to him for all the valuable corrections, suggestions, and constructive help he has provided me with in this work. Without him, these books would have been a lot poorer.

Finally, I would also like to thank the University of Missouri for granting me a research leave during the academic year 2013-2014. This time off enabled me to finish the third edition of this book on time. I spent my leave in Greece.

Contents

1	L^p Spaces and Interpolation	1
1.1	L^p and Weak L^p	1
1.1.1	The Distribution Function	3
1.1.2	Convergence in Measure	6
1.1.3	A First Glimpse at Interpolation	9
	Exercises	11
1.2	Convolution and Approximate Identities	17
1.2.1	Examples of Topological Groups	18
1.2.2	Convolution	19
1.2.3	Basic Convolution Inequalities	21
1.2.4	Approximate Identities	25
	Exercises	30
1.3	Interpolation	33
1.3.1	Real Method: The Marcinkiewicz Interpolation Theorem	33
1.3.2	Complex Method: The Riesz–Thorin Interpolation Theorem	36
1.3.3	Interpolation of Analytic Families of Operators	40
	Exercises	45
1.4	Lorentz Spaces	48
1.4.1	Decreasing Rearrangements	48
1.4.2	Lorentz Spaces	52
1.4.3	Duals of Lorentz Spaces	56
1.4.4	The Off-Diagonal Marcinkiewicz Interpolation Theorem	60
	Exercises	74
2	Maximal Functions, Fourier Transform, and Distributions	85
2.1	Maximal Functions	86
2.1.1	The Hardy–Littlewood Maximal Operator	86
2.1.2	Control of Other Maximal Operators	90

2.1.3	Applications to Differentiation Theory	93
	Exercises	98
2.2	The Schwartz Class and the Fourier Transform	104
2.2.1	The Class of Schwartz Functions	105
2.2.2	The Fourier Transform of a Schwartz Function	108
2.2.3	The Inverse Fourier Transform and Fourier Inversion	111
2.2.4	The Fourier Transform on $L^1 + L^2$	113
	Exercises	116
2.3	The Class of Tempered Distributions	119
2.3.1	Spaces of Test Functions	119
2.3.2	Spaces of Functionals on Test Functions	120
2.3.3	The Space of Tempered Distributions	123
	Exercises	131
2.4	More About Distributions and the Fourier Transform	133
2.4.1	Distributions Supported at a Point	134
2.4.2	The Laplacian	135
2.4.3	Homogeneous Distributions	136
	Exercises	143
2.5	Convolution Operators on L^p Spaces and Multipliers	146
2.5.1	Operators That Commute with Translations	146
2.5.2	The Transpose and the Adjoint of a Linear Operator	150
2.5.3	The Spaces $\mathcal{M}^{p,q}(\mathbf{R}^n)$	151
2.5.4	Characterizations of $\mathcal{M}^{1,1}(\mathbf{R}^n)$ and $\mathcal{M}^{2,2}(\mathbf{R}^n)$	153
2.5.5	The Space of Fourier Multipliers $\mathcal{M}_p(\mathbf{R}^n)$	155
	Exercises	159
2.6	Oscillatory Integrals	161
2.6.1	Phases with No Critical Points	161
2.6.2	Sublevel Set Estimates and the Van der Corput Lemma	164
	Exercises	169
3	Fourier Series	173
3.1	Fourier Coefficients	173
3.1.1	The n -Torus $\mathbf{T}^n$	174
3.1.2	Fourier Coefficients	175
3.1.3	The Dirichlet and Fejér Kernels	178
	Exercises	182
3.2	Reproduction of Functions from Their Fourier Coefficients	183
3.2.1	Partial sums and Fourier inversion	183
3.2.2	Fourier series of square summable functions	185
3.2.3	The Poisson Summation Formula	187
	Exercises	191
3.3	Decay of Fourier Coefficients	192
3.3.1	Decay of Fourier Coefficients of Arbitrary Integrable Functions	193
3.3.2	Decay of Fourier Coefficients of Smooth Functions	195

3.3.3	Functions with Absolutely Summable Fourier Coefficients	200
	Exercises	202
3.4	Pointwise Convergence of Fourier Series	204
3.4.1	Pointwise Convergence of the Fejér Means	204
3.4.2	Almost Everywhere Convergence of the Fejér Means	207
3.4.3	Pointwise Divergence of the Dirichlet Means	210
3.4.4	Pointwise Convergence of the Dirichlet Means	212
	Exercises	214
3.5	A Tauberian theorem and Functions of Bounded Variation	216
3.5.1	A Tauberian theorem	216
3.5.2	The sine integral function	218
3.5.3	Further properties of functions of bounded variation	219
3.5.4	Gibbs phenomenon	221
	Exercises	225
3.6	Lacunary Series and Sidon Sets	226
3.6.1	Definition and Basic Properties of Lacunary Series	227
3.6.2	Equivalence of L^p Norms of Lacunary Series	229
3.6.3	Sidon sets	235
	Exercises	237
4	Topics on Fourier Series	241
4.1	Convergence in Norm, Conjugate Function, and Bochner–Riesz Means	241
4.1.1	Equivalent Formulations of Convergence in Norm	242
4.1.2	The L^p Boundedness of the Conjugate Function	246
4.1.3	Bochner–Riesz Summability	250
	Exercises	253
4.2	A. E. Divergence of Fourier Series and Bochner–Riesz means	255
4.2.1	Divergence of Fourier Series of Integrable Functions	255
4.2.2	Divergence of Bochner–Riesz Means of Integrable Functions	261
	Exercises	270
4.3	Multipliers, Transference, and Almost Everywhere Convergence ...	271
4.3.1	Multipliers on the Torus	271
4.3.2	Transference of Multipliers	275
4.3.3	Applications of Transference	280
4.3.4	Transference of Maximal Multipliers	281
4.3.5	Applications to Almost Everywhere Convergence	285
4.3.6	Almost Everywhere Convergence of Square Dirichlet Means	287
	Exercises	289

4.4	Applications to Geometry and Partial Differential Equations	292
4.4.1	The Isoperimetric Inequality	292
4.4.2	The Heat Equation with Periodic Boundary Condition	294
	Exercises	298
4.5	Applications to Number theory and Ergodic theory	299
4.5.1	Evaluation of the Riemann Zeta Function at even Natural numbers	299
4.5.2	Equidistributed sequences	302
4.5.3	The Number of Lattice Points inside a Ball	305
	Exercises	308
5	Singular Integrals of Convolution Type	313
5.1	The Hilbert Transform and the Riesz Transforms	313
5.1.1	Definition and Basic Properties of the Hilbert Transform . . .	314
5.1.2	Connections with Analytic Functions	317
5.1.3	L^p Boundedness of the Hilbert Transform	319
5.1.4	The Riesz Transforms	324
	Exercises	329
5.2	Homogeneous Singular Integrals and the Method of Rotations	333
5.2.1	Homogeneous Singular and Maximal Singular Integrals	333
5.2.2	L^2 Boundedness of Homogeneous Singular Integrals	336
5.2.3	The Method of Rotations	339
5.2.4	Singular Integrals with Even Kernels	341
5.2.5	Maximal Singular Integrals with Even Kernels	347
	Exercises	353
5.3	The Calderón–Zygmund Decomposition and Singular Integrals . . .	355
5.3.1	The Calderón–Zygmund Decomposition	355
5.3.2	General Singular Integrals	358
5.3.3	L^r Boundedness Implies Weak Type $(1, 1)$ Boundedness . . .	359
5.3.4	Discussion on Maximal Singular Integrals	362
5.3.5	Boundedness for Maximal Singular Integrals Implies Weak Type $(1, 1)$ Boundedness	366
	Exercises	371
5.4	Sufficient Conditions for L^p Boundedness	374
5.4.1	Sufficient Conditions for L^p Boundedness of Singular Integrals	375
5.4.2	An Example	378
5.4.3	Necessity of the Cancellation Condition	379
5.4.4	Sufficient Conditions for L^p Boundedness of Maximal Singular Integrals	380
	Exercises	384
5.5	Vector-Valued Inequalities	385
5.5.1	ℓ^2 -Valued Extensions of Linear Operators	386
5.5.2	Applications and ℓ^r -Valued Extensions of Linear Operators	390

5.5.3	General Banach-Valued Extensions	391
	Exercises	398
5.6	Vector-Valued Singular Integrals	401
5.6.1	Banach-Valued Singular Integral Operators	402
5.6.2	Applications	408
5.6.3	Vector-Valued Estimates for Maximal Functions	411
	Exercises	414
6	Littlewood–Paley Theory and Multipliers	419
6.1	Littlewood–Paley Theory	419
6.1.1	The Littlewood–Paley Theorem	420
6.1.2	Vector-Valued Analogues	426
6.1.3	L^p Estimates for Square Functions Associated with Dyadic Sums	426
6.1.4	Lack of Orthogonality on L^p	431
	Exercises	434
6.2	Two Multiplier Theorems	437
6.2.1	The Marcinkiewicz Multiplier Theorem on $\mathbf{R}$	439
6.2.2	The Marcinkiewicz Multiplier Theorem on $\mathbf{R}^n$	441
6.2.3	The Mihlin–Hörmander Multiplier Theorem on $\mathbf{R}^n$	445
	Exercises	450
6.3	Applications of Littlewood–Paley Theory	453
6.3.1	Estimates for Maximal Operators	453
6.3.2	Estimates for Singular Integrals with Rough Kernels	455
6.3.3	An Almost Orthogonality Principle on L^p	459
	Exercises	461
6.4	The Haar System, Conditional Expectation, and Martingales	463
6.4.1	Conditional Expectation and Dyadic Martingale Differences	464
6.4.2	Relation Between Dyadic Martingale Differences and Haar Functions	465
6.4.3	The Dyadic Martingale Square Function	469
6.4.4	Almost Orthogonality Between the Littlewood–Paley Operators and the Dyadic Martingale Difference Operators	471
	Exercises	474
6.5	The Spherical Maximal Function	475
6.5.1	Introduction of the Spherical Maximal Function	475
6.5.2	The First Key Lemma	478
6.5.3	The Second Key Lemma	479
6.5.4	Completion of the Proof	481
	Exercises	481
6.6	Wavelets and Sampling	482
6.6.1	Some Preliminary Facts	483
6.6.2	Construction of a Nonsmooth Wavelet	485

6.6.3	Construction of a Smooth Wavelet	486
6.6.4	Sampling	490
	Exercises	494
7	Weighted Inequalities	499
7.1	The A_p Condition	499
7.1.1	Motivation for the A_p Condition	500
7.1.2	Properties of A_p Weights	503
	Exercises	511
7.2	Reverse Hölder Inequality for A_p Weights and Consequences	514
7.2.1	The Reverse Hölder Property of A_p Weights	514
7.2.2	Consequences of the Reverse Hölder Property	518
	Exercises	521
7.3	The A_∞ Condition	525
7.3.1	The Class of A_∞ Weights	525
7.3.2	Characterizations of A_∞ Weights	527
	Exercises	530
7.4	Weighted Norm Inequalities for Singular Integrals	532
7.4.1	Singular Integrals of Non Convolution type	532
7.4.2	A Good Lambda Estimate for Singular Integrals	533
7.4.3	Consequences of the Good Lambda Estimate	539
7.4.4	Necessity of the A_p Condition	543
	Exercises	545
7.5	Further Properties of A_p Weights	546
7.5.1	Factorization of Weights	546
7.5.2	Extrapolation from Weighted Estimates on a Single L^{p_0}	548
7.5.3	Weighted Inequalities Versus Vector-Valued Inequalities	554
	Exercises	558
A	Gamma and Beta Functions	563
A.1	A Useful Formula	563
A.2	Definitions of $\Gamma(z)$ and $B(z, w)$	563
A.3	Volume of the Unit Ball and Surface of the Unit Sphere	565
A.4	Computation of Integrals Using Gamma Functions	565
A.5	Meromorphic Extensions of $B(z, w)$ and $\Gamma(z)$	566
A.6	Asymptotics of $\Gamma(x)$ as $x \rightarrow \infty$	567
A.7	Euler's Limit Formula for the Gamma Function	568
A.8	Reflection and Duplication Formulas for the Gamma Function	570
B	Bessel Functions	573
B.1	Definition	573
B.2	Some Basic Properties	573
B.3	An Interesting Identity	576
B.4	The Fourier Transform of Surface Measure on S^{n-1}	577
B.5	The Fourier Transform of a Radial Function on $\mathbf{R}^n$	577