Nonlinear waves in one-dimensional dispersive systems

By

P. L. BHATNAGAR

Nonlinear waves in one-dimensional dispersive systems

BY

P. L. BHATNAGAR

CLARENDON PRESS . OXFORD
1979

Oxford University Press, Walton Street, Oxford OX2 6DP

OXFORD LONDON GLASGOW
NEW YORK TORONTO MELBOURNE WELLINGTON
NAIROBI DAR ES SALAAM CAPE TOWN
KUALA LUMPUR SINGAPORE HONG KONG TOKYO
DELHI BOMBAY CALCUTTA MADRAS KARACHI

British Library Cataloguing in Publication Data

Bhatnagar, P L
Nonlinear waves in one-dimensional dispersive systems. —(Oxford mathematical monographs).
1. Wave motion, Theory of 2. Nonlinear theories
I. Title II. Series
531'.1133 QA927 79-40272

ISBN 0-19-853531-7

Photoset and printed in India by Thomson Press (India) Limited

FOREWORD

Wave motions have the characteristic property that after a signal is observed at one point, a closely related signal may later be observed at a different point. Sometimes the main difference between the two signals is in their *amplitude*, perhaps because the wave's energy is being spread out over a larger area (or focused within a smaller area). Apart from any such change in amplitude, however, various changes in the *shape* of the wave-form are also possible, and great interest is attached to the mechanisms producing these. Most mechanisms causing wave-forms to change shape can be analysed to advantage in one-dimensional systems. This is because the difficulties of analysis are greatly reduced in such systems, without the most crucial features of those mechanisms being suppressed.

This monograph uses this simplification to give a most valuable introduction to the principal mechanisms that act to change wave-forms. These mechanisms include dispersion, dissipation, and nonlinearity, either separately or in various combinations. The analysis includes, furthermore, the study of those remarkable classes of wave-forms for which the distorting effects of *different* mechanisms exactly cancel.

Professor Bhatnagar was outstandingly well qualified to write this monograph. In it, he leads the reader progressively, by simple stages, through an extensive mass of sophisticated modern material within this intriguing field. The resulting book is a quite admirable introduction to its subject.

I had written the above words before the deeply regretted and untimely death of Professor Bhatnagar on 5 October 1976, when the world of applied mathematics suddenly lost one of its most respected figures. After the shock of this great loss had subsided, I felt anxious to ensure that Professor Bhatnagar's last book would receive the wide circulation that it richly merits. I am deeply grateful to Dr. Phoolan Prasad for his excellent work as editor. Applied mathematicians owe him a great debt for helping to make this important text generally available.

JAMES LIGHTHILL

PREFACE

Mehta Research Institute, in collaboration with the Indian Mathematical Society, conducted a four-week course on 'Hyperbolic Systems of Partial Differential Equations and Nonlinear waves' from 17 May to 15 June 1976 for the benefit of the research workers desirous of taking up this fascinating, as well as useful, field of creative activity. The author gave a series of lectures on some aspects of the nonlinear waves. He mainly concentrated on the *steady* solutions of the celebrated model equations that go by the name of Burgers equation and Korteweg–de Vries (KdV) equation, and on soliton interactions, and on the meaning of group velocity for the nonlinear dispersive waves and more briefly touched upon the general equation of evolution of which the KdV equation is a particular case. Out of the many equations of evolution, which have attracted the notice of the outstanding workers in the field during last two decades, choice fell on the two model equations mentioned above simply because the Burgers equation is the simplest model of a diffusive wave and the KdV equation is the simplest model of a dispersive wave. The latter equation has further become important on account of the solitary wave solution which it admits.

The present monograph, more or less faithfully presents the contents of the lectures by the author with the exception of the appendix to Chapter 1 and the two appendices to Chapter 2 which have been included to make it self-contained as far as possible.

The subject of nonlinear waves is being pursued very actively at present and consequently the lectures had to be generally open-ended. It is hoped that the present monograph will provide a necessary background on the techniques and the subject matter.

The author wishes to acknowledge his gratitude to many mathematicians whose work has made this lecture course possible. Among them, he is particularly indebted to Professors M. J. Lighthill, G. B. Whitham, P. D. Lax, R. M. Miura, J. M. Greene, C. S. Gardner, M. D. Kruskal, T. Taniuti, and C. C. Wei as their outstanding contributions to the subject formed the backbone of the course.

Allahabad P. L. B.
September 1976

CONTENTS

NOTE

Professor Bhatnagar was carefully editing the manuscript of this book, making minor changes and correcting errors, when he suddenly passed away on 5 October 1976. It was then that I took up the preparation of the final form of the manuscript, fully aware that had he done it himself, he would have achieved a higher level of perfection.

The publication of this book has been possible due to the keen interest shown by Professor M. J. Lighthill, Lucasian Professor of Mathematics, University of Cambridge. The Oxford University Press, London, coming to a quick decision regarding the publication, has speeded up the bringing out of this book.

It was possible for me to complete the final form of the manuscript within a short time due to the spontaneous help given by Dr. V. G. Tikekar, Dr. Renuka Ravindran, and Dr. Swarnalata Prabhu, who, like me, are also students of Professor Bhatnagar.

Indian Institute of Science
Bangalore
August 1977

PHOOLAN PRASAD

1

LINEAR WAVES

1.1 Introduction

In this chapter we will discuss some important properties of *linear* waves which are governed by linear equations and which are usually described as having *small* amplitudes, which, in reality, means *infinitesimally small* amplitudes. The purpose of including this chapter in a monograph on nonlinear waves is threefold: (i) to introduce necessary terminology; (ii) to focus attention on some important properties which are necessary to understand the nonlinear-wave phenomena which are determined by nonlinear systems of hyperbolic equations; and (iii) to prepare a background against which the properties of linear and nonlinear waves may be compared and contrasted.

We note that in this monograph we shall generally consider waves in one-dimension so that only two independent variables x and t will occur in our discussion. We shall designate x as the spatial coordinate and t as the time coordinate; this sort of specification permits us to use the well-known terminology associated with waves, such as wavelength, wave number, period, frequency, amplitude, wave velocity, group velocity, etc.

1.2. Linear wave equation: wave terminology

Let us start with the celebrated wave equation

$$\phi_{tt} = c^2 \phi_{xx}, \tag{1.1}$$

where ϕ is some property associated with the wave and c^2 is a positive constant. This equation determines the spatial and temporal evolution of ϕ in a homogeneous, isotropic, and conservative system. In fact, we shall define a *wave in a general way as a temporal and spatial evolution of an entity.*

We can write the general solution of (1.1):

$$\phi(x, t) = f(x - ct) + g(x + ct), \tag{1.2}$$

where f and g are arbitrary functions. The first term in (1.2), as we know, represents a *progressive* wave moving in the positive direction of the x-axis with a constant speed c, while the second term represents a progressive wave moving in the negative direction of the x-axis with the same speed c.

The argument $x - ct \equiv p_f$ of the f-wave is called its *phase*. Similarly, $x + ct \equiv p_g$ is called the *phase* of the g-wave. Evidently, p_f is constant in space-time if $\dfrac{dp_f}{dt} = 0$, i.e. if $\dfrac{dx}{dt} = c$. Thus, an observer moving with velocity c with the

f-wave will always notice the same phase of the f-wave and, therefore, the same state of wave motion as indicated by the initial value of f. Similarly, an observer moving with velocity $\frac{dx}{dt} = -c$ along the g-wave will always notice the same phase p_g or the same value of g with which he started. The above statement gives physical meanings to the terms *phase* and *wave velocity*, also called the *phase velocity*.

In a periodic progressive wave (say when f is periodic function of p_f and $g \equiv 0$), a point where ϕ is maximum is called a *crest* and a point where ϕ is minimum is called a *trough*.

In the language of the hyperbolic partial differential equations to which class (1.1) belongs, we say that the eqn (1.1) admits two real *characteristics* in the (x, t)-plane:

$$\left.\begin{aligned} C^+ &: \frac{dx}{dt} = c \\ \text{and} \quad C^- &: \frac{dx}{dt} = -c \end{aligned}\right\} \tag{1.3}$$

Along the first characteristic C^+, $f =$ constant, while along C^-, $g =$ constant. Thus, $f =$ constant and $g =$ constant are the corresponding *compatibility relations*.

We note that the bidirectional propagation of wave represented by (1.1) is not unexpected. The equation is invariant under the transformation:

$$x \to -x, \quad t \to -t. \tag{1.4}$$

The equation is time-reversible and therefore we can study the *future* as well as the *past* of the wave. In contrast, in Chapter 2, we shall discuss *unidirectional* equations of evolution.

Let us now give some particular values to the functions f and g, say

$$\left.\begin{aligned} f(x - ct) &= a \sin(kx - \omega t), c = \frac{\omega}{k} \\ \text{and} \quad g(x + ct) &= 0 \end{aligned}\right\} \tag{1.5}$$

with constant values of ω and k. Then,

$$\phi = a \sin(kx - \omega t) \tag{1.6}$$

represents a periodic progressive wave of *amplitude* a with wave velocity c given by

$$\omega = kc \quad \text{or} \quad \frac{\omega}{k} = c. \tag{1.7}$$

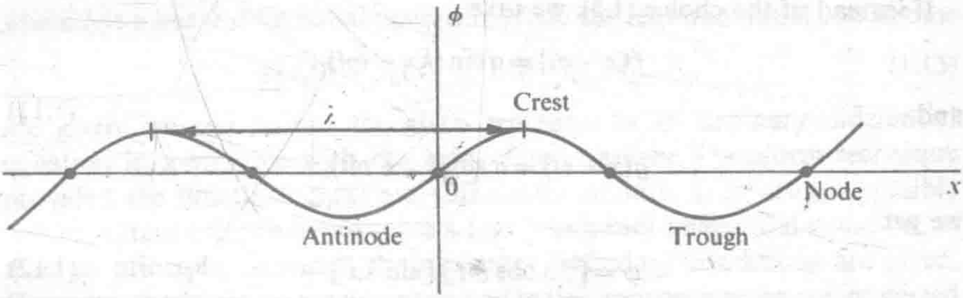

FIG. 1.1. Plot of ϕ against x for a given value of t.

Eqn (1.6) is a solution of the partial differential equation (1.1) satisfying the initial conditions:

$$\phi(x,0) = a \sin(kx), \quad \phi_t(x,0) = -\omega a \cos(kx). \tag{1.8}$$

For a given t, ϕ is sinusoidal in x as indicated in Fig. 1.1.

At any time t, the points $x = (4n+1)\dfrac{\pi}{2k} + ct$, where $n = 0, \pm 1, \pm 2, \ldots$, are where ϕ attains maximum values or crests, and the points $x = (4n+3)\dfrac{\pi}{2k} + ct$, are where it takes minimum values or troughs. The nomenclature, crest and trough, is derived from the geometric shape of the ϕ-profile in Fig. 1.1. The distance between two consecutive crests (or troughs) is called the *wavelength* and is denoted by λ:

$$\lambda = \left\{(4n+5)\frac{\pi}{2k} + ct\right\} - \left\{(4n+1)\frac{\pi}{2k} + ct\right\} = \frac{2\pi}{k}. \tag{1.9}$$

From (1.6) it is clear that k gives the number of waves per unit length (taken here in units of 2π) and hence it is called the wave number. All the points on the ϕ-profile at a given time, whose *abscissae* differ by integral multiples of λ, have the same phase.

At a given point with abscissa, say x_1, ϕ oscillates with respect to t with *period*

$$P = \frac{2\pi}{\omega}. \tag{1.10}$$

$\omega = \dfrac{2\pi}{\mathrm{P}}$ is called the (angular) *frequency* of the wave and denotes the number of waves passing through a fixed point per unit time (taken here in units of 2π).

If instead of the choice (1.5), we take

$$f(x-ct)=a\sin(kx-\omega t)$$

and (1.11)

$$g(x+ct)=a\sin(kx+\omega t),$$

we get

$$\phi=[2a\cos\omega t][\sin kx] \tag{1.12}$$

so that we can study the variations of ϕ with respect to x and t independently of each other. This choice evidently corresponds to the following initial conditions for ϕ:

$$\phi(x,0)=2a\sin kx,\quad \phi_t(x,0)=0. \tag{1.13}$$

The points $x=\dfrac{n\pi}{k}$, where $\phi=0$ at *all* times are called *nodes* of the wave. The points $x=(2n+1)\dfrac{\pi}{2k}$, where ϕ attains extreme values are called *antinodes*. The solution (1.12) has been obtained by the superposition of two sinusoidal progressive waves of equal amplitude, wavelength, and frequency, but moving in opposite directions. Except at the nodes, the quantity ϕ oscillates in t with period P, the amplitude at the antinodes is maximum and equal to $2a$ which is clearly equal to the sum of the amplitudes of the component f- and g-waves.

Since there is no communication in the form of energy or momentum transfer between the waves separated by nodes, the wave form represented by eqn (1.12) is called the *standing* wave. The concept of nodes and antinodes are peculiar to the standing wave.

From the above description, it is clear that a solution to (1.1) under certain circumstances represents a *progressive wave* and under certain other circumstances represents a *standing wave*. We also know that the *transverse* wave in an elastic wire stretched taut, in which the motions of various points of the wire are at right-angles to the direction of wave propagation, is represented mathematically by the eqn (1.1). This equation also represents the *longitudinal* sound wave in air in which the air molecules vibrate about their mean positions in the direction of wave propagation.

1.3. General linear equation, dispersion relation

We have introduced the above terms with the help of a specific equation, called the standard linear wave equation. Let us now consider a general linear partial differential equation in two independent variables x and t:

$$L[\phi]=0, \tag{1.14}$$

where L is a linear differential operator. When the required initial conditions:

$$\phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), \ldots, \tag{1.15}$$

are given, we can reduce the given equation to an ordinary differential equation in x-variable with the help of the Laplace Transform technique provided the functions $\phi_i(x)$ are sufficiently smooth. It is always possible (under certain conditions) to solve a linear ordinary differential equation, at least in principle, provided the necessary boundary conditions are given. However, at present we are not interested in this approach as we are interested in a general discussion of eqn (1.14). The equation being linear, we can build up its general solution by superposition of its various Fourier components. Consequently, let us substitute

$$\phi = a \exp\{\mathrm{i}(kx - \omega t)\} \tag{1.16}$$

in eqn (1.14) in which we assume now that the independent variables x and t do not appear explicitly and the equation is homogeneous. This substitution removes all derivatives with respect to t and x: $\frac{\partial}{\partial t} \to -\mathrm{i}\omega$, $\frac{\partial}{\partial x} \to \mathrm{i}k$ and reduces it to the following relation:

$$D(\omega, k; A_l) = 0, \tag{1.17}$$

where A_l are the parameters occurring in eqn (1.14). Eqn (1.17) is the *dispersion relation* which determines the frequency ω of the wave in terms of the wave number k and the parameters A_l. We shall write (1.17) *formally* as

$$\omega = \omega(k; A_l). \tag{1.18}$$

The number of roots of eqn (1.17) depends on the degree n of this algebraic equation in ω. Clearly, n is equal to the highest order of t-derivative in eqn (1.14). We consider each root separately as each one of them gives a separate wave, called a *mode*.

Let us consider a general root

$$\omega = \omega(k), \tag{1.19}$$

where we have supressed the dependence of ω on A_l as in the present discussion they do not play any specific role. The corresponding Fourier component is

$$\phi(x,t) \propto \exp[\mathrm{i}\{kx - \omega(k)t\}].$$

The temporal evolution of ϕ depends on the nature of $\omega(k)$. The following cases arise:

(i) when $\omega(k)$ is real, this Fourier component represents a harmonic wave.

(ii) When $\omega(k)[= \mathrm{i}\omega_2(k)]$ is pure imaginary,

$$\phi(x,t) \propto \exp(\mathrm{i}kx).\exp\{\omega_2(k)t\}$$

so that we get a nonpropagating standing wave. If $\operatorname{Im}\omega(k) > 0, \phi$ becomes unbounded exponentially with t; if $\operatorname{Im}\omega(k) < 0, \phi$ decays exponentially with t. Thus, in the former case, we deal with a growing (amplifying) wave, while in the latter case we deal with a decaying (attenuating) wave. In the former case, the initial disturbance imposed on the system grows without bound and the system is said to be *unstable* with respect to the particular mode under consideration; in the latter case, the system is said to be *stable* with respect to the mode.

(iii) Let $$\omega(k) = \omega_1(k) + i\omega_2(k),$$

where ω_1 and ω_2 are real. Here

$$\phi \propto \exp[i\{kx - \omega_1(k)t\}] . \exp(\omega_2(k)t)$$

so that when $\omega_2 = \operatorname{Im}\omega < 0$, the wave is harmonic with exponentially decaying amplitude. The system is stable with respect to the mode in this case. When $\operatorname{Im}\omega > 0$, the wave is harmonic with exponentially growing amplitude. The system is unstable with respect to the mode but Eddington (1926) calls this type of instability *overstability* because it is provoked by restoring forces so strong as to overshoot the corresponding position on the other side of the equilibrium. This sets up an oscillation of increasing amplitude.

The above discussion of the dispersion relation brings out clearly its importance in determining the response of the system to an imposed disturbance, which is assumed to be of infinitesimally small amplitude initially.

The dispersion relation also provides a basis for another classification of waves. Let us *assume* that eqn (1.19) determines a real value of ω for each value of $k: 0 \leq k < \infty$. If $\dfrac{\partial^2 \omega}{\partial k^2} = \omega''(k) \not\equiv 0$, the wave is said to be *dispersive*, when $\omega''(k) \equiv 0$, it is said to be *non-dispersive*. It also introduces a new characteristic velocity, called the *group velocity* denoted by $V_g = \omega'(k)$.

The dispersion relation provides still another basis for classification of waves. When eqn (1.19) determines a complex value for ω, the wave is said to be *diffusive*; when ω is real, the wave is said to be *non-diffusive*. The diffusive waves are associated with attenuation of the amplitudes with time due to certain dissipative mechanisms present in the system.

1.4. Dispersive waves: group velocity

Having introduced the terms group velocity, and dispersive and non-dispersive waves mathematically in an abstract manner, we shall now give them some physical meaning.

Group velocity

Let us consider the superposition of two harmonic waves which differ very slightly in their frequencies and wave numbers, but have the same amplitude:

$$\phi_1(x,t) = a\cos(kx - \omega t), \tag{1.20}$$

$$\phi_2(x,t) = a\cos\{(k+\delta k)x - (\omega + \delta\omega)t\}. \tag{1.21}$$

As a result

$$\phi = \phi_1 + \phi_2 = \left[2a\cos\left\{\frac{1}{2}(x\delta k - t\delta\omega)\right\}\right]\cos\left\{\left(k + \frac{\delta k}{2}\right)x - \left(\omega + \frac{\delta\omega}{2}\right)t\right\}, \tag{1.22}$$

which is the familiar expression for *beats*.

ϕ oscillates with frequency $\omega + \frac{1}{2}\delta\omega$ which is slightly different to ω and has a wavelength which is also slightly different to $\lambda = 2\pi/k$. The effective amplitude

$$A = 2a\cos\{\tfrac{1}{2}(x\delta k - t\delta\omega)\} \tag{1.23}$$

varies slowly with period $\dfrac{4\pi}{\delta\omega}$ and wavelength $\dfrac{4\pi}{\delta k}$ between the sum of the amplitudes of the component waves and zero. Since $\delta\omega$ and δk are small, the period and wavelength of A are both large.

As a result of constructive and destructive interference, the ϕ-profile along both time and space axes appear as a series of periodically repeating groups as shown in Fig. 1.2. Each group consists of a number of waves.

The surface over which the group amplitude remains constant is defined by the equation

$$x\delta k - t\delta\omega = \text{Constant}. \tag{1.24}$$

From eqn (1.24) it follows that the groups themselves are propagated with velocity

$$\frac{dx}{dt} = \frac{\delta\omega}{\delta k} = \omega'(k) \text{ (in the limit } \delta k \to 0), \tag{1.25}$$

where the prime denotes differentiation with respect to k. Therefore, the group velocity V_g is given by

$$V_g = \omega'(k)$$

$$= \frac{\text{difference in frequencies of component waves}}{\text{difference in wave numbers of component waves}}$$

The above discussion imparts physical meaning to the group velocity.

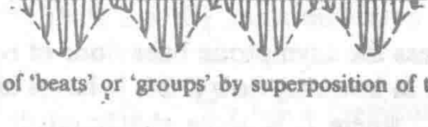

FIG. 1.2. Formation of 'beats' or 'groups' by superposition of two harmonic waves.

We have earlier defined the phase velocity V_p through the relation

$$V_p = \omega/k.$$

Dispersive and non-dispersive waves

In general, both phase velocity and group velocity are functions of the wave number. We can easily check that when $\omega''(k) \not\equiv 0$, V_g is different from V_p and depends on k so that the waves of different wavelengths travel with different group velocities. Let us consider a disturbance initiated at $x=0$ at time $t=0$ which consists of a superposition of a number of wavelengths. Since the components of the disturbance with different wave numbers travel with different velocities, after some time the disturbance will be spread over a certain length which increases with time. In this situation we say that the wave has undergone *dispersion*. It is also clear that along the wave-train the wave number varies slowly.

When $\omega''(k) \equiv 0$, both the phase velocity and group velocity coincide and there is no separation of waves of different wave numbers. In this case the wave is non-dispersive.

Example

On substituting (1.16) in (1.1) we get the dispersion relation $\omega = \pm ck$, so that here $V_g = V_p = \pm c$ and the wave represented by (1.1) is non-diffusive and non-dispersive.

1.5 General solution of the linear-wave equation

We have so far considered individual Fourier components of a linear wave. We can obtain the general solution of the equation by superposing these individual Fourier components:

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k)\exp[\mathrm{i}\{kx - \omega(k)t\}]\mathrm{d}k, \tag{1.26}$$

where $\omega = \omega(k)$ is the function of the wave number and the parameters of the problem as determined by the dispersion relation and the *spectrum function* $A(k)$ takes care of the initial condition. In principle, we can always construct the spectrum function in a given problem though at times it may be very tedious to do so. The solution (1.26) corresponds to the initial condition

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)\exp(\mathrm{i}kx)\mathrm{d}k, \tag{1.27}$$

which is the Fourier integral for $\phi(x,0)$ and consequently, given $\phi(x,0)$, $A(k)$ can be evaluated.

We shall now discuss the asymptotic behaviour of eqn (1.26) as $t \to \infty$. In fact we are interested in knowing how (1.26) behaves after the lapse of large time, i.e., when $t \gg t_c$, where t_c is some characteristic time, like period P, associated with the wave. The simplest method for obtaining this asymptotic value is the method of *steepest descent* or the *saddle-point method* because it demands the least possible details. (In Appendix I at the end of this chapter, we have briefly described this method. See also Jeffreys and Jeffreys (1946), and Dennery and Krzywicki (1967).)

We write (1.26) in the form

$$\phi(x,t)=\int_{-\infty}^{\infty} A(k)\exp\{\mathrm{i}t\chi(k)\}\,\mathrm{d}k, \tag{1.28}$$

where the phase function $\chi(k)$ is given by

$$\chi(k)=\frac{x}{t}k-\omega(k). \tag{1.29}$$

We assume that $\chi(k)$ is analytic in the complex k-plane for a fixed value of x/t. In most of the physical problems of interest, this assumption is always valid.

A saddle point is defined as a point where the phase function $\chi(k)$ attains a stationary value. Therefore, in the present case, the saddle points are given by

$$\left.\frac{\partial\chi(k)}{\partial k}\right|_{(x/t)\text{ fixed}}=0, \tag{1.30a}$$

i.e. by

$$\omega'(k)=x/t, \text{ provided } \omega''(k)\neq 0. \tag{1.30b}$$

On solving for k, we get the saddle points

$$k_i=k_i(x/t). \tag{1.30c}$$

Since the path of integration is along the real line, it is sufficient to consider only real saddle points k_i.

Corresponding to the saddle point k_i, the saddle-point methods gives the following asymptotic value for $\phi(x,t)$

$$\phi(x,t)\simeq\frac{\sqrt{(2\pi)}A(k_i)\exp\{\mathrm{i}t\chi(k_i)+\mathrm{i}\alpha\}}{\{t|\chi''(k_i)|\}^{1/2}} \tag{1.31a}$$

$$=\frac{\sqrt{(2\pi)}A(k_i)\exp[\mathrm{i}\{k_ix-\omega(k_i)t\}+\mathrm{i}\alpha]}{\{t|\omega''(k_i)|\}^{1/2}} \tag{1.31b}$$

as $t\to\infty$,

where

$$\alpha=\frac{\pi}{4}, \text{ if } \omega''(k_i)<0, \text{ i. e. if } \chi \text{ has a minimum value at } k_i$$

$$=-\frac{\pi}{4}, \text{ if } \omega''(k_i)>0, \text{ i.e. if } \chi \text{ has a maximum value at } k_i. \tag{1.32}$$

Every other saddle point contributes to the value of $\phi(x,t)$ similarly and thus, taking all the saddle points, m in number, into account, we have

$$\phi(x,t) \simeq \sum_{i=1}^{m} \frac{A(k_i)\sqrt{(2\pi)}\exp\left[\mathrm{i}\{k_i x - \omega(k_i)t\} - \frac{\mathrm{i}\pi}{4}\operatorname{sgn}\omega''(k_i)\right]}{\{t|\omega''(k_i)|\}^{1/2}}, \tag{1.33}$$

where

$$\left.\begin{aligned} \operatorname{sgn}\omega''(k_i) &= -1, \quad \text{if } \omega''(k_i) < 0 \\ &= 1, \quad \text{if } \omega''(k_i) > 0 \end{aligned}\right\}. \tag{1.34}$$

The asymptotic expression (1.31b) for $\phi(x,t)$ appears surprising in many ways:

(i) it represents a locally harmonic wave which is not uniform in the sense that, in view of (1.30c), k_i and $\omega(k_i)$ vary with x and t (through the combination x/t), in spite of the fact that the initial state of the wave was not harmonic;

(ii) ultimately, i.e. when $t \gg P$, a phase difference is introduced which is equal to $\frac{\pi}{4}$ if the group velocity $\omega'(k)$ decreases with k and is equal to $-\frac{\pi}{4}$ if the group velocity $\omega'(k)$ increases with k; and

(iii) when $\omega''(k_i) \neq 0$, the amplitude $\bar{A}(t)$ of the wave given by

$$\bar{A}(t) = \frac{\sqrt{(2\pi)}\,A(k_i)}{\{t|\omega''(k_i)|\}^{1/2}} \tag{1.35}$$

decreases inversely as the square root of t over distances and times of the order of x and t themselves as seen from the following discussion of the relative changes in k with x and t. Assuming that $\omega''(k_i) \neq 0$ and rewriting (1.30b), we have for a saddle point k

$$x = \omega'(k)t. \tag{1.36}$$

On differentiating partially with respect to x and t, we can easily show that

$$\frac{k_x}{k} = \frac{\omega'(k)}{k\omega''(k)}\frac{1}{x} = O\left(\frac{1}{x}\right) \tag{1.37}$$

and

$$\frac{k_t}{k} = -\frac{\omega'(k)}{k\omega''(k)}\frac{1}{t} = O\left(\frac{1}{t}\right). \tag{1.38}$$

Since we are considering large values of time ($t \gg P$) and large distances ($x \gg \lambda$), the above expressions predict relative changes of $O(1)$ only over times