Lecture Notes on Knot Invariants

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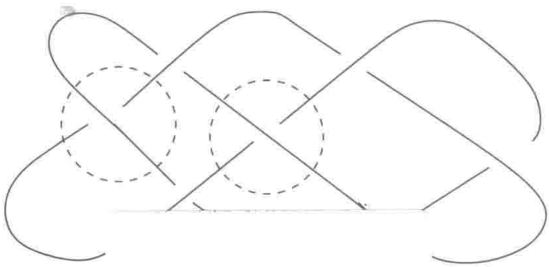

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Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Li, Weiping, 1963–
Lecture notes on knot invariants / by Weiping Li (Southwest Jiaotong University, China & Oklahoma State University, USA).
pages cm
Includes bibliographical references and index.
ISBN 978-9814675956 (hardcover : alk. paper) -- ISBN 978-9814675963 (softcover : alk. paper)
1. Knot theory. 2. Braid theory. 3. Low-dimensional topology. 4. Alexander ideals. 5. Knot polynomials. I. Title. II. Title: Knot invariants.
QA612.2.L5 2015
514'.2242--dc23
2015021070

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore by B & Jo Enterprise Pte Ltd

Lecture Notes on Knot Invariants

To my family

Diana, Oliver, and Xiaoli

Preface

Knot theory represents a rich mixture of many branches of mathematics, including topology, algebra, and geometry. It is also rich in its interactions with chemistry, physics and, more generally, technology. One of the fundamental problems in knot theory is the question of how to tell when two knots are really different. The main idea is to assign some sort of invariants to knots, so that if the invariants for two knots are different, then so are the knots. It would be best to have a complete set of easily computable invariants, so that we could go the other way, saying when the knots are equal - a much harder question.

There are many books that deal with knot invariants from various points of view. We do not try to cover everything here, but rather emphasize basic calculation skills and particular invariants. The reader will be introduced to this beautiful subject by means of some classical knot invariants, which have geometrical or topological origins aspects. We will explain the Jones polynomial from the original approach using braids and representations of Hecke algebras. We will understand the Casson-Lin invariants via representations of braids.

By proceeding in this way, we are forced, unfortunately, to skip many other interesting topics in knot theory. On the other hand, we feel strongly that it is better for students and readers to learn how to get their hands on the topics, rather than to know some fancy words. All the materials presented in this book can be checked by fairly direct methods, or at least can be followed by some basic steps toward the understandings. Of course, some basic knowledge of algebra and topology is required.

We start from the basic knot presentations, their equivalence classes, and the well-known Reidemeister moves. In Chapter 2, we introduce braids and their relationship to knots and links. Some classical invariants of knots and links are given and discussed in Chapter 3. By no means is this a complete list. There are many textbooks dealing with knots and their invariants. We choose to concentrate on the original definition of the Jones polynomial and the proof of Tait's conjectures. The Casson-type invariants of knots are constructed from braids and their representations. It is one of the typical and important problems in knot theory to find a geometrical or topological interpretation of the Jones polynomial which is constructed from algebra or to find a combinatorial interpretation of the Casson-type invariants which is constructed from geometry and topology.

This book is based on lecture notes that were originally prepared for the Chinese Graduate Summer School at Sichuan University in July–August, 2000. Some parts were also taught at Oklahoma State University. I would like to thank Xiao-Song Lin for many helpful conversations and communications, which helped me to understand the subject better, and to thank my collaborators Weiping Zhang and Qingxue Wang for many stimulating discussions. The students at the Chinese Graduate Summer School and Oklahoma State University also provided useful comments.

I would like to thank World Scientific Publishing Company for publishing these lecture notes, and also Rebecca Fu for her patience and help. Thanks also go to my students Zhili Chen, Xiaowei Yang and Bin Xie for various assistance in preparing this book.

Last but not least I want to thank my family for being there all the time.

Weiping Li

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Chapter 1

Basic knots, links and their equivalences

1.1 Definitions and equivalences

Definition 1.1.1. Let S^1 be a unit circle. L is a *link* in a closed 3-manifold Y if there is an embedding map $L : S^1 \bigsqcup \cdots \bigsqcup S^1 \to Y$ from disjoint unit circles to Y. Let $\mu(L)$ be the number of S^1-components in $S^1 \bigsqcup \cdots \bigsqcup S^1$. If $\mu(L) = 1$, then L is called a *knot* in Y.

Remark 1.1.2. (1) We do not distinguish knots and links as images of embeddings (Im (L)) or the embeddings themselves (L). There are higher dimensional knots (embedding $S^k \to M^{K+2}$ for $k \geq 2$) in [Rolfsen, 1976] which we are not taking into consideration in this book. We restrict ourselves only to knots and links in 3-manifolds in this book.

(2) One can consider knots and links in different topological categories. If the embedding L is (C^∞) smooth, then L is a smooth link or smooth knot. If L is piecewise linear, then L is a PL link (PL knot) (sometimes called polygonal link or polygonal knot).

Definition 1.1.3. Two knots or links L and $L^{'}$ are equivalent if there is a homeomorphism $h : Y \to Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} S^1 \bigsqcup \cdots \bigsqcup S^1 & \xrightarrow{L} & Y \\ \Big\downarrow = & & \Big\downarrow h \\ S^1 \bigsqcup \cdots \bigsqcup S^1 & \xrightarrow{L^{'}} & Y. \end{array} \tag{1.1}$$

If Y is an oriented closed 3-manifold and h is an orientation-preserving homeomorphism, then L and $L^{'}$ are oriented equivalent.

Remark 1.1.4. (1) Two equivalent knots and links have the property that there is a homeomorphism identifying their images in the 3-manifold Y.

(2) The map h is dependent upon the category one wants to work with. If two knots or links are C^{∞}, then the equivalence is given by a diffeomorphism $h : Y \to Y$ in Definition 1.1.3. Similarly for PL equivalence, h must be a piecewise linear homeomorphism.

Exercise 1.1.5. Prove that the relation defined in Definition 1.1.3 is an equivalent relation.

Let $\mathcal{L}(Y)$ ($\mathcal{L}^+(Y)$) be the space of (orientation-preserving) equivalence classes of links and knots in Y (oriented) under the above equivalent relation.

Definition 1.1.6. Two links (or knots) L and $L^{'}$ are *ambient isotopy* if there exists a map $H : [0,1] \times Y \to Y$ such that (i) $H(0,\cdot) = Id_Y$, (ii) $H(t,\cdot) : Y \to Y$ is a homeomorphism for each $t \in [0,1]$, and (iii) $H_1 \circ L = L^{'}$.

Any two ambient isotopy links are certainly equivalent. But it is not true in general that any pair of equivalent links or knots is ambient isotopic. It depends on the mapping class group of Y. Let $\mathcal{L}_{ai}(Y)$ ($\mathcal{L}^+_{ai}(Y)$) be the space of equivalence classes of (orientation-preserving) ambient isotopy knots or links in Y (oriented). Thus we have $\mathcal{L}_{ai}(Y) \subset \mathcal{L}(Y)$ and $\mathcal{L}^+_{ai}(Y) \subset \mathcal{L}^+(Y)$.

Proposition 1.1.7. *If the mapping class group of Y has only one path-connected component, then $\mathcal{L}_{ai}(Y) = \mathcal{L}(Y)$; If the orientation-preserving mapping class group of oriented Y has only one path-connected component, then $\mathcal{L}^+_{ai}(Y) = \mathcal{L}^+(Y)$.*

Proof: There is always a path-connected component of the identity map. By the hypothesis, any equivalent (orientation-preserving) homeomorphism h can be connected by a (orientation-preserving) homeomorphism path from Id_Y. Thus we obtain $\mathcal{L}(Y) \subset \mathcal{L}_{ai}(Y)$ and $\mathcal{L}^+(Y) \subset \mathcal{L}^+_{ai}(Y)$. Therefore results follow. □

Example 1.1.8. If $Y = \mathbb{R}^3$ (noncompact oriented) is the Euclidean space of real 3-tuples, then the orthogonal group $O(3)$ (3×3 matrices with determinant ± 1 form a compact submanifold of $\mathbb{R}^9$ of dimension 3) is a deformation retract of $Diff(\mathbb{R}^3)$ (the diffeomorphic mapping class group of $\mathbb{R}^3$). Note that $O(3)$ has two components, the component of the identity is the subgroup $SO(3)$ of orthogonal matrices of determinant 1. Therefore $Diff_+(\mathbb{R}^3)$ (orientation-preserving diffeomorphisms of $\mathbb{R}^3$) has only one path-connected component. By Proposition 1.1.7,

$$\mathcal{L}^+_{ai}(\mathbb{R}^3) = \mathcal{L}^+(\mathbb{R}^3).$$

The orientation-preserving diffeomorphism equivalent knots or links in $\mathbb{R}^3$ are also C^∞-ambient isotopy to each other.

Example 1.1.9. If $Y = S^3$ is the (compact oriented) 3-sphere, then $\mathit{Diff}_+(S^3)$ has only one path-connected component. Let $f : B^{n+1} \to B^{n+1}$ be an orientation-preserving diffeomorphism, where B^{n+1} is the $(n+1)$-dimensional ball with a restriction map $r : B^{n+1} \to \partial B^{n+1} = S^n$. Hence the restriction $f|_{\partial B^{n+1}} : S^n \to S^n$ is an orientation-preserving diffeomorphism. Such a restriction map r defines a group homomorphism from $Diff_+(B^{n+1})$ to $Diff_+(S^n)$. Let G_n be the image of this group homomorphism. If $g \in Diff_+(S^n)$ is isotopic to Id_{S^n}, then g can be extended to an orientation-preserving diffeomorphism of B^{n+1}, i.e., $g \in G_n$. So G_n is the path-connected component of the identity of S^n. Let $\Gamma_n = Diff_+(S^n)/G_n$ be the quotient group (always Abelian). There is a short exact sequence

$$0 \to Diff_+(B^{n+1}) \xrightarrow{r} Diff_+(S^n) \to \Gamma_n \to 0.$$

In fact every Γ_n is important in classifying differential structures. The set of diffeomorphism classes of oriented differential structures on S^n forms a group H_n under connected sum. $H_n \cong \Gamma_n$ except perhaps for $n = 4$. It is still an open question whether there is an exotic smooth structure on S^4 (*smooth Poincaré conjecture for 4-dimensional sphere* S^4). It is known that the Γ_n's are finite for all n except the case $n = 4$, by the computation of Kervaire and Milnor. The first nontrivial group is $\Gamma_7 \cong \mathbb{Z}_{28}$ by Milnor [1956]. $\Gamma_2 = 0$ by Smale [1959] and Munkres [1960]. $\Gamma_3 = 0$ is an interesting and difficult Morse-theoretic proof by Cerf [1974]. Therefore we have

$$\mathcal{L}^+_{ai}(S^3) = \mathcal{L}^+(S^3).$$

Example 1.1.10. For $Y = \mathbb{R}P^3$, the set of 1-dimensional subspaces of $\mathbb{R}^4$ through the origin can be identified with the real projective space $\mathbb{R}P^3$. Hence S^3 is a 2-sheeted covering space of $\mathbb{R}P^3$. By the result of Cerf [1974], $\mathit{Diff}(S^3)$ has two components that are path-connected to Id_{S^3} and $-Id_{S^3}$ respectively. Under the identification in $\mathbb{R}P^3 = S^3/\pm Id$, the mapping class group of $\mathbb{R}P^3$ has only one path-connected component containing $Id_{\mathbb{R}P^3}$. Hence we obtain $\mathcal{L}(\mathbb{R}P^3) = \mathcal{L}_{ai}(\mathbb{R}P^3)$ by Proposition 1.1.7.

1.2 Polygonal (PL), smooth (C^∞)-links and knots in $\mathbb{R}^3$

It is convenient to present knots and links in $\mathbb{R}^3$ through their perpendicular projections on a plane. Call a link segment of a polygonal link L in $\mathbb{R}^3$ an *edge* of L, and an end point of an edge a *vertex* of L. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be an orthogonal projection , and $p(L)$ be the image of L in the projected plane.

Definition 1.2.1. A point $x \in p(L)$ is a multiple point if the cardinality of the set $p^{-1}(x) \cap L$ is greater than or equal to 2.

If the cardinality $|p^{-1}(x) \cap L| = n$, then x is called an n-multiple point, or a point of order n. Any 2-multiple point x is also called a double point.

The projection should be generic in the sense that (i) $n \leq 2$ for any n-multiple point, (ii) the number of double points in $p(L)$ is finite, and (iii) every vertex of the link L has order 1.

Definition 1.2.2. A projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ is a regular projection for L in $\mathbb{R}^3$ if for every $x \in p(L)$, then $o(x)$ (the order of x) has the following property:

(1) For any $x \in p(L)$, $o(x) \leq 2$;
(2) The set $C_p(L) = \{x \in p(L) : o(x) = 2\}$ is finite;
(3) For any $x \in C_p(L)$, $p^{-1}(x) \cap L$ does not contain any vertex of L.

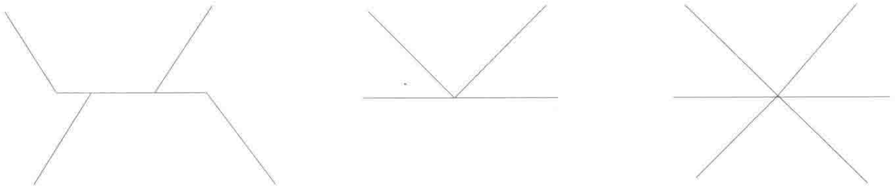

Fig. 1.1 Irregular multiple points

The first condition for regular projections is to rule out any multiple point other than double points, the second is to present the link with only finite crossings, the third is to avoid non-transversal double points. Hence any multiple point of any regular projection has a local image which looks like a letter X. The multiple points in Figure 1.1 are not contained in any regular projection of Definition 1.2.2.

Proposition 1.2.3. *Any polygonal link L has a regular projection.*

Proof: Any projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ has the same image for two parallel planes, and it is completely determined by a fixed point and perpendicular direction to the projection plane. If we fix the point to be the origin, then the space of projections is the space of straight lines through the origin.

This is the two-dimensional projective plane $\mathbb{R}P^2$. There is one-to-one correspondence between projections $\mathbb{R}^3 \to \mathbb{R}^2$ with $p(0,0,0) = (0,0)$ and straight lines in $\mathbb{R}^3$ considered as elements in $\mathbb{R}P^2$.

Let S be the set of non-regular projections. Then it corresponds to a set of straight lines through the origin which do not satisfy Definition 1.2.2. Let S_1 be the subset of non-regular projections that have order 2 non-transverse points (Definition 1.2.2 (3) invalid). Let S_2 be the subset of non-regular projections which have order ≥ 3 multiple points (Definition 1.2.2 (1) and (2) invalid). We have $S_1 \cup S_2 \subset S$. Any non-regular projection must have either that the vertex is a double point (projection in S_1) or that a multiple point has order ≥ 3 (projection in S_2). Therefore $S = S_1 \cup S_2$. The set S_1 consists of finite line segments in $\mathbb{R}P^2$, and the set S_2 consists of finite many curve segments of second order. Hence S is a one-dimensional subset of $\mathbb{R}P^2$ (see [Crowell and Fox, 1977] for more details). The result follows. □

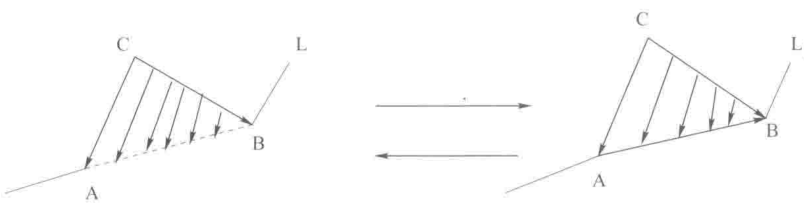

Fig. 1.2 An elementary move

Definition 1.2.4. (1) An elementary move of a polygonal link L is a replacement of L by a new link L_1 in the following way: there is a triangle ABC formed by three vertices which do not intersect any other point of L, two edges AC and BC are replaced by AB in a link L_1 (a disk remove) (see Figure 1.2).

(2) Two links L and L' are equivalent if they can be joined by a finite sequence of links $L, L_1, \cdots, L_n = L'$ in which each subsequent link is obtained from the previous one by an elementary move or its inverse.

Exercise 1.2.5. Show that the relation between two links defined in Definition 1.2.4 is indeed an equivalence relation.

Exercise 1.2.6. Let $P\mathcal{L}(\mathbb{R}^3)$ be the set of equivalence classes of polygonal links in $\mathbb{R}^3$. Prove that the equivalent relation defined by elementary moves is the same as the relation defined in Definition 1.1.3. Hence $P\mathcal{L}(\mathbb{R}^3) = \mathcal{L}(\mathbb{R}^3)$.

Let $f : S^1 \to \mathbb{R}^3$ be a smooth embedding, i.e., $f(t) = (f_1(t), f_2(t), f_3(t))$ has the following properties: (a) each $f_i : S^1 \to \mathbb{R}$ is a smooth function, and (b) $df : T_{t_0}S^1 \to T_{f(t_0)}\mathbb{R}^3$ is injective for every $t_0 \in S^1$. The property (b) is the same as saying that the linear transformation $df(t_0)$ from $\mathbb{R} = T_{t_0}S^1$ to $\mathbb{R}^3 = T_{f(t_0)}\mathbb{R}^3$ has zero kernel, where

$$df(t_0) \cdot v = \begin{pmatrix} f_1'(t_0) \\ f_2'(t_0) \\ f_3'(t_0) \end{pmatrix} \cdot v, \qquad v \in T_{t_0}S^1.$$

Exercise 1.2.7. Verify $f : [0, 2\pi] \to \mathbb{R}^3$ is a smooth knot, where

$$f(\tau) = ((2 + \cos 3\tau)\cos 2\tau, (2 + \cos 3\tau)\sin 2\tau, \sin 3\tau).$$

In fact this is a smooth parametrization of a trefoil knot.

Theorem 1.2.8. *[Burde and Zieschange, 1985, Proposition 1.10] There is a bijective map from the equivalence classes of polygonal links in $\mathbb{R}^3$ to the equivalence classes of smooth links in $\mathbb{R}^3$.*

Remark 1.2.9. (1) For links in general 3-manifolds, we have to add an extra Riemannian metric, and replace each edge by a geodesic. Theorem 1.2.8 is also true for the 3-manifold S^3.

For a closed compact oriented 3-manifold $Y (\neq R^3)$, we can use the Riemannian metric g_Y and geodesics to define a PL-link in Y, where an edge of the link is a geodesic with respect to the metric g_Y. Similarly we have elementary moves and its corresponding relations. Hence the set of equivalence classes is $P\mathcal{L}_{g_Y}(Y)$, the PL-links in the 3-manifold (Y, g_Y). So $P\mathcal{L}_{g_Y}(Y) \subset \mathcal{L}_{g_Y}(Y)$.

Does there exist a metric g_Y such that $\mathcal{L}_{g_Y}(Y) \subset P\mathcal{L}_{g_Y}(Y)$? This is equivalent to asking whether there exists a metric g_Y such that the embedding $L : S^1 \bigsqcup S^1 \bigsqcup \cdots \bigsqcup S^1 \to Y$ can be represented by finitely many geodesics. Conjecturally $\mathcal{L}_{g_Y}(Y) \subset P\mathcal{L}_{g_Y}(Y)$ if there is a uniform lower bound for the injective radius.

(2) For usual knots and links in $\mathbb{R}^3$, the PL theory and C^∞ theory provide the same result. This is why people often use these two approaches interchangeably.