DE GRUYTER

Zoran Stanić

REGULAR GRAPHS

A SPECTRAL APPROACH

SERIES IN DISCRETE MATHEMATICS AND APPLICATIONS 4

Zoran Stanić

Regular Graphs

A Spectral Approach

DE GRUYTER

Mathematics Subject Classification 2010
05C50, 05C12, 05C76, 05C81, 05B05, 05B20, 65F15, 68R10, 94B05

Author
Prof. Dr. Zoran Stanić
University of Belgrade
Faculty of Mathematics
Studentski trg 16
11000 Belgrade
Serbia
zstanic@math.rs

ISBN 978-3-11-035128-6
e-ISBN (PDF) 978-3-11-035134-7
e-ISBN (EPUB) 978-3-11-038336-2
Set-ISBN 978-3-11-035135-4
ISSN 2195-5557

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Printing and binding: CPI books GmbH, Leck
♾ Printed on acid-free paper
Printed in Germany

MIX
Papier aus verantwortungsvollen Quellen
FSC® C083411
www.fsc.org

www.degruyter.com

Zoran Stanić
Regular Graphs

De Gruyter Series in Discrete Mathematics and Applications

Volume 4

Preface

This book has been written to be of use to scientists working in the theory of graph spectra. It may also attract the attention of graduate students dealing with the same subject area as well as all those who use graph spectra in their research, in particular, computer scientists, chemists, physicists or electrical engineers.

In the entire book we investigate exactly one class of graphs (regular) by using exactly one approach (spectral), that is we study eigenvalues and eigenvectors of a graph matrix and their interconnections with the structure and other invariants. Regular graphs appear in numerous sources. In particular, they can probably be encountered in all the books concerning graph theory. An intriguing phenomenon is that, in many situations, regular graphs are met as extremal, exceptional or unique graphs with a given property. More tangibly, an inequality is attained for a regular graph or a claim holds unless a graph under consideration is regular or it does hold exactly for regular graphs. In other words, depending on a given problem they fluctuate from one amplitude to another, which is making them probably the most investigated and the most important class of graphs, and simultaneously motivating us to consider the theory of graph spectra through the prism of regular graphs. There is also a relevance of regular graphs in the study of other discrete structures such as linear spaces or block designs. A final motivation for this book is a remarkable significance of regular graphs in branches of computer science, chemistry, physics and other disciplines that consider processes dealing with graph-like objects with high regularity and many symmetries.

The rapid development of graph theory in last decades caused an inability to cover all relevant results even if we restrict ourselves to a singular approach, so we must say that this material represents a narrowed selection of (in most cases, known) results. Our intention was to include classical results concerning the spectra of regular graphs along with recent developments. Occasionally, possible applications are indicated. The terminology and notation are taken from our previous book [290]. This book was written to be as self-contained as possible, but we assume a decent mathematical knowledge, especially in algebra and the theory of graph spectra.

Here is a brief outline of the contents. Chapter 1 is preparatory. In Chapter 2 we focus on basic properties of the spectra of regular graphs. In Chapter 3 we pay attention to some particular types of regular graph. Graph products, walk-regular graphs, different subclasses of distance-regular graphs, regular graphs of line systems and various block designs occupy the central place of this chapter. In Chapter 4 we consider the problem of determining regular graphs that satisfy fixed spectral constraints. There, we mostly deal with regular graphs with bounded least or second largest eigenvalue, those with integral spectrum or a comparatively small number of distinct eigenvalues. Cospectrality of regular graphs is also considered. Chapter 5 deals with a single subclass of regular graphs called expanders. Expanders have an enormous number

DOI: 10.1515/9783110351347-202

of surprising properties making them relevant in mathematics and other contexts. We consider the three different approaches to these graphs, their constructions and applications in the coding theory. For almost all standard square matrices associated with graphs, the spectrum of one of them contains full information about the spectrum of the remaining ones whenever a graph under consideration is regular, which means that in the major part of the research there is no difference which matrix we are dealing with. An exception is the distance matrix which is the subject of Chapter 6.

Observing the table of contents, the reader may notice that expanders explored in Chapter 5 can be considered as a part of Chapter 3. The reasons for separating them in a single chapter are the quite different approach they require and the size of the presented material. Many results presented in Chapter 4 are based on those of Chapter 3 which motivated us to set these two chapters next to one another.

All chapters are divided into sections and some sections are divided into subsections. The reader will also notice the existence of paragraphs that occur irrespectively of the above hierarchy.

Each of the Chapters 2–6 contains theoretical results, comments (including additional explanations or possible applications), examples and exercises. Some of the results presented in these chapters can be found in other books concerning the theory of graph spectra. In particular, the books [49, 87, 89] cover some parts of Chapter 2. Some results of Chapter 3 can be found in [48] (especially those related to distance-regular graphs), [145] (strongly regular graphs, Kneser graphs, line systems), [94] (line systems, star complements) or [62] (block designs). The book [94] also covers a smaller part of Chapter 4. Chapter 5 is partially covered by [113]. For details that are not presented here, we recommend some of these books.

The author is grateful to Tamara Koledin who read parts of the manuscript and gave valuable suggestions, and to Sebastian Cioabă, Dragoš Cvetković, Edwin R. van Dam, Willem H. Haemers, Tamara Koledin again and Peter Rowlinson for permissions to reproduce some of their proofs without significant change. Finally, Nancy Christ, Apostolos Damialis and Nadja Schedensack on behalf of the publisher helped with editing the book by answering a lot of questions, which is much appreciated.

Last but not least, the author apologizes in advance for possible misprints or computational errors as well as for the inconvenience these might cause to the reader.

Contents

List of Figures

1 Introduction

Here we give a survey of the main graph-theoretic terminology, notation and necessary results. The presentation is separated into three sections. In the first two we deal with graph structure and give some observations including statistical data on regular graphs. In the third section we focus on the adjacency matrix and the corresponding spectrum.

Since all parts of this chapter can be found in numerous sources, by the assumption that the reader is familiar with most of the concepts presented, we orientate to a brief but very clear and intuitive exposition. More details are given in the introductory chapter of our previous book [290].

1.1 Terminology and notation

Vertices and edges

Let G be a finite undirected simple graph (so, without loops or multiple edges). We denote its set of vertices (resp. edges) by V or $V(G)$ (resp. E or $E(G)$). In addition, we assume that $|V| \neq \emptyset$. The quantities $n = |V|$ and $m = |E|$ are called the *order* and the *size* of G, respectively. Two vertices u and v are *adjacent* (or *neighbours*) if they are joined by an edge. In this case we write $u \sim v$ and say that the edge uv is incident with vertices u and v. Similarly, two edges are *adjacent* if they are incident with a common vertex.

The set of neighbours (or the *neighbourhood*) of a vertex v is denoted $N(v)$. The *closed neighbourhood* of v is denoted $N[v]$ $(= \{v\} \cup N(v))$.

Two graphs G and H are said to be *isomorphic* if there is a bijection between sets of their vertices which respects adjacencies. If so, then we write $G \cong H$. Observe that the graphs illustrated in Figure 1.1 are isomorphic. In particular, an *automorphism* of a graph is an isomorphism to itself.

The *degree* d_v of a vertex v is the number of edges incident with it. The minimal and the maximal vertex degrees in a graph are denoted δ and Δ, respectively. A vertex of degree 1 is called an *endvertex* or a *pendant* vertex.

We say that a graph G is *regular* of degree r if all its vertices have degree r. If so, then G is referred to as r-regular. In particular, a 3-regular graph is called a *cubic graph*.

The *complete graph* with n vertices K_n is the unique $(n-1)$-regular graph. Similarly, the *cocktail party* $CP(n)$ is the unique $(2n-2)$-regular graph with $2n$ $(n \geq 1)$ vertices. The complete graph K_1 is called the *trivial graph*.

The *r-dimensional cube* Q_r is an r-regular graph of order 2^r with the vertex set $V(Q_r) = \{0, 1\}^r$ (all possible binary r-tuples) in which two vertices are adjacent if they differ in exactly one coordinate (see Figure 1.1).

DOI: 10.1515/9783110351347-001

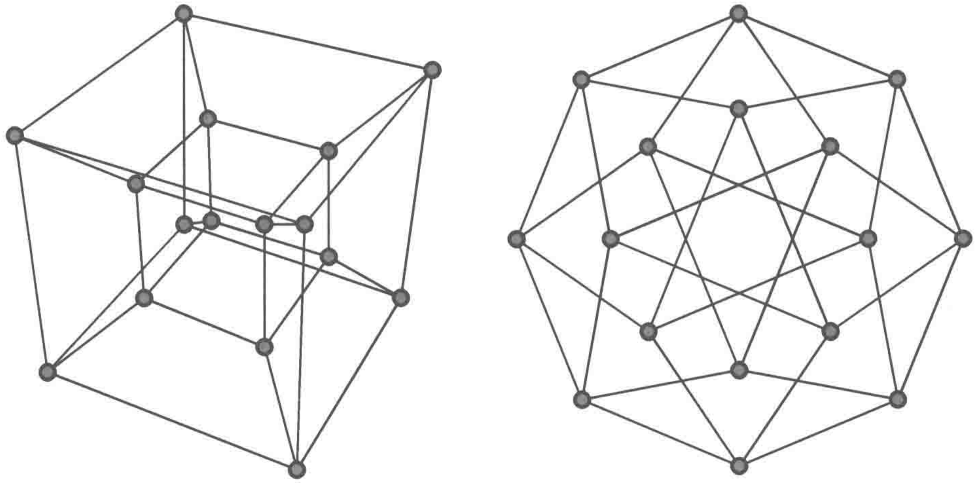

Fig. 1.1. Two illustrations of the 4-dimensional cube.

The *edge degree* of an edge uv is defined as equal to $d_u + d_v - 2$. In other words, it is the number of edges adjacent to uv. Similarly to the above, a graph is *edge-regular* of degree s if all its edges have degree s.

A *matching* in a graph is a set of edges such that any vertex in a graph is incident with at most one edge of a matching. We say that a matching is *perfect* if every vertex is incident with an edge from the matching.

Induced subgraphs and subgraphs

An *induced subgraph* of a graph G is any graph H obtained by deleting some vertices (together with their edges incident). A graph G is *H-free* if it does not contain H as an induced subgraph. A *subgraph* of G is any graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, if $V(H) = V(G)$ then H is called a *spanning subgraph* of G.

If $S \subset V(G)$, then we write $G[S]$ to denote the induced subgraph of G with vertex set S in which two vertices are adjacent if and only if they are adjacent in G. For short, $G - v$ (resp. $G - e$) designates the induced subgraph (resp. subgraph) obtained by deleting a single vertex v (resp. edge e). We call it a *vertex-deleted* (resp. *edge-deleted*) subgraph.

If S and T are disjoint subsets of $V(G)$, then $m(S)$ and $m(S, T)$ stand for the number of edges in $G[S]$ and the number of edges with one end in S and the other in T, respectively.

A *clique* is any complete induced subgraph of a graph G. The *clique number* ω is the number of vertices in the largest clique of G. Similarly, a *co-clique* is any induced subgraph without edges. The vertices of a co-clique make an *independent set* of vertices of G, while the number of vertices in the largest independent set is called the *independence number* and denoted α.

Walks and connectivity

A *k-walk* (or simply, *walk*) in a graph G is a sequence of alternate vertices and edges $v_1, e_1, v_2, e_2, \dots, e_{k-1}, v_k$ such that, for $1 \leq i \leq k-1$, the vertices v_i and v_{i+1} are distinct ends of the edge e_i. A walk is closed if v_1 coincides with v_k. The number of k-walks is denoted w_k, while the number of closed k-walks starting at vertex v is denoted $w_k(v)$. The *length* of a walk is equal to the number of its edges.

Remark 1.1.1. We suggest to the reader to remember that in this book the parameter k denotes the number of vertices in a k-walk, while its length is then equal to $k-1$.

A *path* is a walk in which all vertices are mutually distinct. A graph which is itself a path with n vertices is denoted P_n. There is a special case of a path consisting of a single vertex (i.e., $P_1 \cong K_1$). A vertex of minimal degree in P_n is called an *end* of P_n. By inserting an edge between the ends of P_n $(n \geq 3)$, we get a *cycle* C_n. The cycle C_3 is known as a *triangle*, C_4 is known as a *quadrangle*, and so on. Since all paths and cycles are particular walks, the *length* of a path or a cycle is equal to the number of its edges. A graph with n vertices is *Hamiltonian* if it contains C_n as a spanning subgraph. Any such cycle is referred to as a *Hamiltonian cycle*.

We say that a graph G is *connected* if every two (not necessary distinct) vertices are the ends of at least one path in G. Otherwise, we say that G is *disconnected*. Maximal connected induced subgraphs of a disconnected graph are referred to as its *components*. A component consisting of a single vertex is called an *isolated vertex* of a graph, while a graph consisting entirely of (at least two) isolated vertices is called *totally disconnected*.

A *tree* is a connected graph that does not contain any cycle (as a subgraph). Obviously, the number of edges m of any tree equals $n-1$.

The *vertex* (resp. *edge*) *connectivity* c_v (resp. c_e) of a connected graph is the minimal number of vertices (resp. edges) whose removal results in a trivial or disconnected graph.

The *distance* $d(u, v)$ between (not necessary distinct) vertices u and v belonging to the same component is the length of the shortest path between u and v. The *diameter* of a connected graph G is defined by $\mathcal{D} = \max\{d(u, v) : u, v \in V(G)\}$ (i.e., it is the longest distance between two vertices of G). The *girth* g is the length of the shortest cycle induced in G.

Colouring and bipartite graphs

A graph G is *k-colourable* if its vertices can be properly coloured (i.e., in such a way that two adjacent vertices are coloured by different colours) by k colours. The *chromatic number* χ is the minimal value of k such that G is k-colourable.

A graph G is called *bipartite* if its chromatic number is 1 or 2. Obviously, $\chi(G) = 1$ holds if and only if $E(G) = \emptyset$. The vertex set of a bipartite graph can be partitioned into two parts (or colour classes) X and Y in such a way that every edge of G joins a

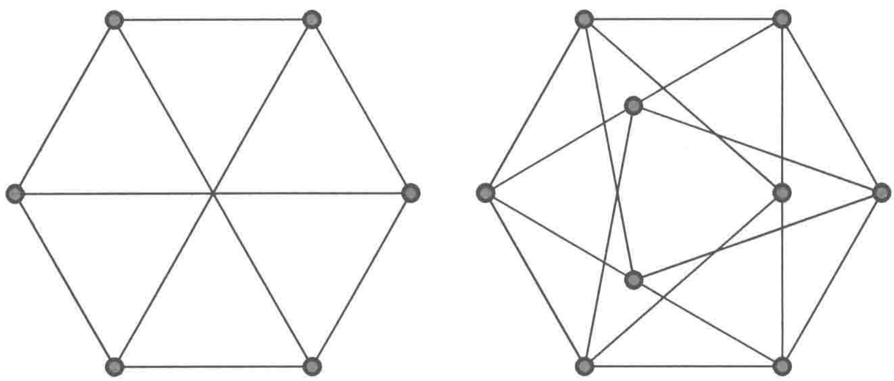

Fig. 1.2. Cubic graph $\overline{2C_3}$ and its 4-regular line graph.

vertex in X with a vertex in Y. Observe that there is a unique partition whenever G is connected (on up to interchanging the roles of X and Y). A graph is called *complete bipartite* if every vertex in one part is adjacent to every vertex in the other part. If $|X| = n_1$ and $|Y| = n_2$, the complete bipartite graph is denoted K_{n_1,n_2}. In particular, if at least one of the parameters is equal to 1, it is called a *star*. A vertex with maximal degree in the star $K_{1,n-1}$ is called the *centre*. We use $K_{n,n}^-$ to denote the graph obtained by removing a perfect matching from $K_{n,n}$.

More general, a *k-partite* graph is a graph whose set of vertices can be partitioned into k parts such that no two vertices in the same part are adjacent. If there is an edge between every pair of vertices belonging to different parts, the graph is referred to as a *complete k-partite* or, if k is suppressed, a *complete multipartite*.

A graph is called *bipartite semiregular* if it is bipartite and the vertices belonging to the same part have equal degree. If the corresponding vertex degrees are r and s, then the graph is referred to as a bipartite (r, s)-semiregular.

Certain operations

For two graphs G and H we define $G \cup H$ to be their disjoint union. We also use kG to denote the disjoint union of k copies of G. The *join* $G \nabla H$ is the graph obtained by inserting an edge between every vertex of G and every vertex of H. This operation may be applied to a sequence of graphs $G_1, G_2, \ldots, G_k$. In that case we usually write $G_1 \nabla G_2 \nabla \cdots \nabla G_k$.

The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set as G, in which any two distinct vertices are adjacent if and only if they are non-adjacent in G. Observe that if G is disconnected, then $\overline{G}$ must be connected. If G is bipartite with parts X and Y, then its *bipartite complement* is the bipartite graph $\overline{\overline{G}}$ with the same parts having the edge between X and Y exactly where G does not.

A *multigraph* includes the possible existence of loops or multiple edges. All previous numerical invariants can be defined for multigraphs in a similar way.