Third Edition

COMPUTATIONAL CONTINUUM MECHANICS

Ahmed A. Shabana

$$\mathbf{Q}_g = \int_V \mathbf{S}^{\mathrm{T}} \begin{bmatrix} 0 & -g & 0 \end{bmatrix}^{\mathrm{T}} dV \qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{J}^T\mathbf{J} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{J}_o^{-1T}\left(\mathbf{J}_e^T\mathbf{J}_e\right)\mathbf{J}_o^{-1} - \mathbf{I}\right) \qquad \mathbf{J} = \frac{\partial \mathbf{r}}{\partial \mathbf{X}} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right) = \mathbf{J}_e\mathbf{J}_o^{-1}$$

$$s_{ij} = \sum_{k=1}^{3} t_{ijk} b_k = t_{ij1} b_1 + t_{ij2} b_2 + t_{ij3} b_3 \qquad t_{ijk} = \sum_{l=1}^{3} f_{ijkl} v_l = f_{ijk1} v_1 + f_{ijk2} v_2 + f_{ijk3} v_3$$

WILEY

An updated and expanded edition of the popular guide to basic continuum mechanics and computational techniques

This updated third edition of the popular reference covers state-of-the-art computational techniques for basic continuum mechanics modeling of both small and large deformations. Approaches to developing complex models are described in detail, and numerous examples are presented demonstrating how computational algorithms can be developed using basic continuum mechanics approaches.

The integration of geometry and analysis for the study of the motion and behaviors of materials under varying conditions is an increasingly popular approach in continuum mechanics, and absolute nodal coordinate formulation (ANCF) is rapidly emerging as the best way to achieve that integration. At the same time, simulation software is undergoing significant changes which will lead to the seamless fusion of CAD, finite element, and multibody system computer codes in one computational environment. *Computational Continuum Mechanics, Third Edition* is the only book to provide in-depth coverage of the formulations required to achieve this integration.

- Provides detailed coverage of the absolute nodal coordinate formulation (ANCF), a popular new approach to the integration of geometry and analysis
- Provides detailed coverage of the floating frame of reference (FFR) formulation; a popular, well-established approach for solving small deformation problems
- Supplies numerous examples of how complex models have been developed to solve an array of real-world problems
- Covers modeling of both small and large deformations in detail
- Demonstrates how to develop computational algorithms using basic continuum mechanics approaches

Computational Continuum Mechanics, Third Edition is designed to function equally well as a text for advanced undergraduates and first-year graduate students and as a working reference for researchers, practicing engineers, and scientists working in computational mechanics, bio-mechanics, computational biology, multibody system dynamics, and other fields of science and engineering using the general continuum mechanics theory.

Ahmed A. Shabana, PhD, is University Distinguished Professor and the Richard and Loan Hill Professor of Engineering at the University of Illinois at Chicago. Professor Shabana is the author of several books and serves on the Editorial Board of several journals. He served as the Chair of the ASME Design Engineering Division, the Founding Chair of the ASME Technical Committee on Multibody Systems and Nonlinear Dynamics, and the Founding Chair of the ASME International Conference of Multibody Systems, Nonlinear Dynamics, and Control. Professor Shabana is a Fellow of the American Society of Mechanical Engineers (ASME) and a Fellow of the Society of Automotive Engineering (SAE International). He has received several awards, including the Humboldt Prize, the Fulbright Research Scholar Award, the ASME D'Alembert Award, and Honorary Doctorate, Honorary Professorship, and Best Paper Awards, as well as several teaching and research awards from the University of Illinois at Chicago.

Cover Design: Wiley
Cover Images: (Gas Tank) © scanrail/Gettyimages; (Tire) © Rellas/Gettyimages; (Water) © molotovcoketail/Gettyimages; (Wave) © Shannon Stent/iStockphoto; (Leaf Springs) © ABBPhoto/Gettyimages

ISBN 978-1-119-29321-7

www.wiley.com

WILEY

Also available as an e-book

9 781119 293217

Shabana

COMPUTATIONAL CONTINUUM MECHANICS

Third Edition

WILEY

COMPUTATIONAL CONTINUUM MECHANICS
THIRD EDITION

AHMED A. SHABANA
Richard and Loan Hill Professor of Engineering
University of Illinois at Chicago
Chicago, Illinois, USA

WILEY

This edition first published 2018

First edition published 2008 by Cambridge University Press
Second edition published 2012 by Cambridge University Press

Registered Office(s)
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

Editorial Office
The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, UK

For details of our global editorial offices, customer services, and more information about Wiley products visit us at www.wiley.com.

Wiley also publishes its books in a variety of electronic formats and by print-on-demand. Some content that appears in standard print versions of this book may not be available in other formats.

Library of Congress Cataloging-in-Publication Data:

Names: Shabana, Ahmed A., 1951- author.
Title: Computational continuum mechanics / by Ahmed A. Shabana.
Description: Third edition. | Hoboken, NJ, USA : Wiley, [2018] | Includes
bibliographical references and index. |
Identifiers: LCCN 2017032713 (print) | LCCN 2017035575 (ebook) | ISBN
9781119293231 (ePDF) | ISBN 9781119293200 (ePUB) | ISBN 9781119293217
(cloth)
Subjects: LCSH: Continuum mechanics. | Engineering mathematics.
Classification: LCC QA808.2 (ebook) | LCC QA808.2 .S46 2018 (print) | DDC
531–dc23
LC record available at https://lccn.loc.gov/2017032713

Cover design by Wiley
Cover Images: (Gas Tank) © scanrail/Gettyimages; (Tire) © Rellas/Gettyimages; (Water) © molotovcoketail/Gettyimages; (Wave) © Shannon Stent/iStockphoto; (Leaf Springs) © ABBPhoto/Gettyimages

Typeset in 10/12pt TimesLTStd by SPi Global, Chennai, India
Printed and bound in Malaysia by Vivar Printing Sdn Bhd

10 9 8 7 6 5 4 3 2 1

COMPUTATIONAL
CONTINUUM MECHANICS

PREFACE

Nonlinear continuum mechanics is one of the fundamental subjects that form the foundation of modern computational mechanics. The study of the motion and behavior of materials under different loading conditions requires understanding of basic, general, and nonlinear kinematic and dynamic relationships that are covered in continuum mechanics courses. The finite element method, on the other hand, has emerged as a powerful tool for solving many problems in engineering and physics. The finite element method became a popular and widely used computational approach because of its versatility and generality in solving large-scale and complex physics and engineering problems. Nonetheless, the success of using the continuum-mechanics-based finite element method in the analysis of the motion of bodies that experience general displacements, including arbitrary large rotations, has been limited. The solution to this problem requires resorting to some of the basic concepts in continuum mechanics and putting the emphasis on developing sound formulations that satisfy the principles of mechanics. Some researchers, however, have tried to solve fundamental formulation problems using numerical techniques that lead to approximations. Although numerical methods are an integral part of modern computational algorithms and can be effectively used in some applications to obtain efficient and accurate solutions, it is the opinion of many researchers that numerical methods should only be used as a last resort to fix formulation problems. Sound formulations must be first developed and tested to make sure that these formulations satisfy the basic principles of mechanics. The equations that result from the use of the analytically correct formulations can then be solved using numerical methods.

This book is focused on presenting the nonlinear theory of continuum mechanics and demonstrating its use in developing nonlinear computer formulations that can be used in the large displacement dynamic analysis. To this end, the basic concepts used in continuum mechanics are first presented and then used to develop nonlinear general finite element formulations for the large displacement analysis. Two nonlinear finite element dynamic formulations will be considered in this book. The first is a general large-deformation finite element formulation, whereas the second is a formulation that can be used efficiently to solve small-deformation problems that characterize very and moderately stiff structures. In this latter case, an elaborate method for eliminating the unnecessary degrees of freedom must be used in order to be able to efficiently obtain a numerical solution. An attempt has been made to present the materials in a clear and systematic manner with the assumption that the reader has only basic knowledge in matrix and vector algebra as well as basic knowledge of dynamics. The book is designed for a course at the senior undergraduate and first-year graduate level. It can also be used as a reference for researchers and practicing engineers and scientists who are working in the areas of computational mechanics, biomechanics, computational biology, multibody system dynamics, and other fields of science and engineering that are based on the general continuum mechanics theory.

In **Chapter 1**, matrix, vector, and tensor notations are introduced. These notations will be repeatedly used in all chapters of the book, and therefore, it is necessary that the reader reviews this chapter in order to be able to follow the presentation in subsequent chapters. The polar decomposition theorem, which is fundamental in continuum and computational mechanics, is also presented in this chapter. D'Alembert's principle and the principle of virtual work can be used to systematically derive the equations of motion of physical systems. These two important principles are discussed and the relationship between them is explained. The use of a finite dimensional model to describe the continuum motion is also discussed and the procedure for developing the discrete equations of motion is outlined. The principles of momentum and principle of work and energy are presented, and the problems associated with some of the finite element formulations that violate these analytical mechanics principles are discussed. Chapter 1 also provides a discussion on the definitions of the gradient vectors that are used in continuum mechanics to define the strain components.

In **Chapter 2**, the general kinematic displacement equations of a continuum are developed and used to define the strain components. The Green–Lagrange strains and the Almansi or Eulerian strains are introduced. The Green–Lagrange strains are defined in the reference configuration, whereas the Almansi or Eulerian strains are defined in the current deformed configuration. The relationships between these strain components are established and used to shed light on the physical meaning of the strain components. Other deformation measures as well as the velocity and acceleration equations are also defined in this chapter. The important issue of objectivity that must be considered when large deformations and inelastic formulations are used is discussed. The equations that govern the change of volume and area, the conservation of mass, and examples of deformation modes are also presented in this chapter.

Forces and stresses are discussed in **Chapter 3**. Equilibrium of forces acting on an infinitesimal material element is used to define the Cauchy stresses, which are used to develop the partial differential equations of equilibrium. The transformation of the stress components and the symmetry of the Cauchy stress tensor are among the topics discussed in this chapter. The virtual work of the forces due to the change of the shape of the continuum is defined. The deviatoric stresses, stress objectivity, and energy balance equations are also discussed in Chapter 3.

The definition of the strain and stress components is not sufficient to describe the motion of a continuum. One must define the relationship between the stresses and strains using the constitutive equations that are discussed in **Chapter 4**. In Chapter 4, the generalized Hooke's law is introduced and the assumptions used in the definition of homogeneous isotropic materials are outlined. The principal strain invariants and special large-deformation material models are discussed. The linear and nonlinear viscoelastic material behavior is also discussed in Chapter 4.

Nonlinear finite element formulations are discussed in **Chapters 5** and **6**. Two formulations are discussed in these two chapters. The first is a large-deformation finite element formulation, which is discussed in **Chapter 5**. This formulation, called the absolute nodal coordinate formulation (ANCF), is based on a continuum mechanics theory and employs position gradients as coordinates. It leads to a unique displacement and rotation fields and imposes no restrictions on the amount of rotation or deformation within the finite element. The absolute nodal coordinate formulation has some unique features that distinguish it from other existing large-deformation finite element formulations: it leads to a constant mass matrix; it leads to zero centrifugal and Coriolis forces; it automatically satisfies the principles of mechanics; it correctly describes an arbitrary rigid-body motion including finite rotations; and it can be used to develop several beam, plate, and shell elements that relax many of the assumptions used in classical theorems. When using ANCF finite elements, no distinction is made between plate and shell elements since shell geometry can be systematically obtained using the nodal coordinates in the reference configuration.

Clearly, large-deformation finite element formulations can also be used to solve small-deformation problems. However, it is not recommended to use a large-deformation finite element formulation to solve a small-deformation problem. Large-deformation formulations do not exploit some particular features of small-deformation problems, and therefore, such formulations can be very inefficient in the solution of stiff and moderately stiff systems. The development of an efficient small-deformation finite element formulation that correctly describes an arbitrary rigid-body motion requires the use of more elaborate techniques in order to define a local linear problem without compromising the ability of the method to describe large-displacement, small-deformation behavior. The finite element floating frame of reference (FFR) formulation, widely used in the analysis of small deformations, is discussed in **Chapter 6**. This formulation allows eliminating high-frequency modes that do not have a significant effect on the solution, thereby leading to a lower-dimension dynamic model that can be efficiently solved using numerical and computer methods.

Although finite element (FE) formulations are based on polynomial representations, the polynomial-based geometric representation used in computer-aided design (CAD) methods cannot be converted exactly to the kinematic description used in many existing FE formulations. For this reason, converting a CAD model to an FE mesh can be costly and time-consuming. CAD software systems use computational geometry methods such as B-spline and Non-Uniform Rational B-Splines (NURBS). These methods can describe accurately complex geometry. The relationship between these CAD geometry methods and the FE formulations presented in this book are discussed in **Chapter 7**. As explained in Chapter 7, modeling modern engineering and physics systems requires the successful integration of computer-aided design and analysis (I-CAD-A) by developing an efficient interface between CAD systems and analysis tools or by developing a new mechanics based CAD/analysis system.

In many engineering applications, plastic deformations occur due to excessive forces and impact as well as thermal loads. Several plasticity formulations are presented in **Chapter 8**. First, a one-dimensional theory is used in order to discuss the main concepts and solution procedures used in the plasticity analysis. The theory is then generalized to the three-dimensional analysis for the case of small strains. Large strain nonlinear plasticity formulations as well as the J_2 flow theory are among the topics discussed in Chapter 8.

I would like to thank many students and colleagues with whom I worked for several years on the subject of flexible body dynamics. I was fortunate to collaborate with excellent students and colleagues who educated me in this important field of computational mechanics. In particular, I would like to thank my doctorate students, Bassam Hussein, Luis Maqueda, Mohil Patel, Brian Tinsley, and Liang Wang, who provided solutions for several of examples and figures presented in several chapters of the book. I would also like to thank my family for their help, patience, and understanding during the time of preparing this book.

AHMED A. SHABANA
Chicago, IL
2016

CONTENTS

CHAPTER 1

INTRODUCTION

Matrix, vector, and tensor algebras are often used in the theory of continuum mechanics in order to have a simpler and more tractable presentation of the subject. In this chapter, the mathematical preliminaries required to understand the matrix, vector, and tensor operations used repeatedly in this book are presented. Principles of mechanics and approximation methods that represent the basis for the formulation of the kinematic and dynamic equations developed in this book are also reviewed in this chapter. In the first two sections of this chapter, matrix and vector notations are introduced and some of their important identities are presented. Some of the vector and matrix results are presented without proofs with the assumption that the reader has some familiarity with matrix and vector notations. In Section 3, the summation convention, which is widely used in continuum mechanics texts, is introduced. This introduction is made despite the fact that the summation convention is rarely used in this book. Tensor notations, on the other hand, are frequently used in this book and, for this reason, tensors are discussed in Section 4. In Section 5, the *polar decomposition theorem*, which is fundamental in continuum mechanics, is presented. This theorem states that any nonsingular square matrix can be decomposed as the product of an orthogonal matrix and a symmetric matrix. Other matrix decompositions that are used in computational mechanics are also discussed. In Section 6, D'Alembert's principle is introduced, while Section 7 discusses the virtual work principle. The finite element method is often used to obtain finite dimensional models of continuous systems that in reality have infinite number of degrees of freedom. To introduce the reader to some of the basic concepts used to obtain finite dimensional models, discussions of approximation methods are included in Section 8. The procedure for developing the discrete equations of motion is outlined in Section 9, while the principle of conservation of momentum and the principle of work and energy are discussed in Section 10. In continuum mechanics,

Computational Continuum Mechanics, Third Edition. Ahmed A. Shabana.

the gradients of the position vectors can be determined by differentiation with respect to different parameters. The change of parameters can lead to the definitions of strain components in different directions. This change of parameters, however, does not change the coordinate system in which the gradient vectors are defined. The effect of the change of parameters on the definitions of the gradients is discussed in Section 11.

1.1 MATRICES

In this section, some identities, results, and properties from matrix algebra that are used repeatedly in this book are presented. Some proofs are omitted, with the assumption that the reader is familiar with the subject of linear algebra.

Definitions

An $m \times n$ matrix $\mathbf{A}$ is an ordered rectangular array, which can be written in the following form:

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{1.1}$$

where a_{ij} is the ijth element that lies in the ith row and jth column of the matrix. Therefore, the first subscript i refers to the row number and the second subscript j refers to the column number. The arrangement of Equation 1 shows that the matrix $\mathbf{A}$ has m rows and n columns. If $m = n$, the matrix is said to be *square*; otherwise, the matrix is said to be *rectangular*. The *transpose* of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix, denoted as $\mathbf{A}^{\mathrm{T}}$, which is obtained from $\mathbf{A}$ by exchanging the rows and columns, that is, $\mathbf{A}^{\mathrm{T}} = (a_{ji})$.

A *diagonal matrix* is a square matrix whose only nonzero elements are the diagonal elements, that is, $a_{ij} = 0$ if $i \neq j$. An *identity* or *unit matrix*, denoted as $\mathbf{I}$, is a diagonal matrix that has all its diagonal elements equal to one. The *null* or *zero matrix* is a matrix that has all its elements equal to zero. The *trace* of a square matrix $\mathbf{A}$ is the sum of all its diagonal elements, that is,

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} \tag{1.2}$$

This equation shows that $\mathrm{tr}(\mathbf{I}) = n$, where $\mathbf{I}$ is the identity matrix and n is the dimension of the matrix.

A square matrix $\mathbf{A}$ is said to be *symmetric* if

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}, \quad a_{ij} = a_{ji} \tag{1.3}$$

A square matrix is said to be *skew symmetric* if

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}, \quad a_{ij} = -a_{ji} \tag{1.4}$$