Australian Mathematical Society Lecture Series 27

Orthogonal Polynomials and Painlevé Equations

Walter Van Assche

There are a number of intriguing connections between Painlevé equations and orthogonal polynomials, and this book is one of the first to provide an introduction to these. Researchers in integrable systems and nonlinear equations will find the many explicit examples where Painlevé equations appear in mathematical analysis very useful. Those interested in the asymptotic behavior of orthogonal polynomials will also find the description of Painlevé transcendents and their use for local analysis near certain critical points helpful to their work.Rational solutions and special function solutions of Painlevé equations are worked out in detail, with a survey of recent results and an outline of their close relationship with orthogonal polynomials. Exercises throughout the book help the reader to get to grips with the material.

The author is a leading authority on orthogonal polynomials, giving this work a unique perspective on Painlevé equations.

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Van Assche

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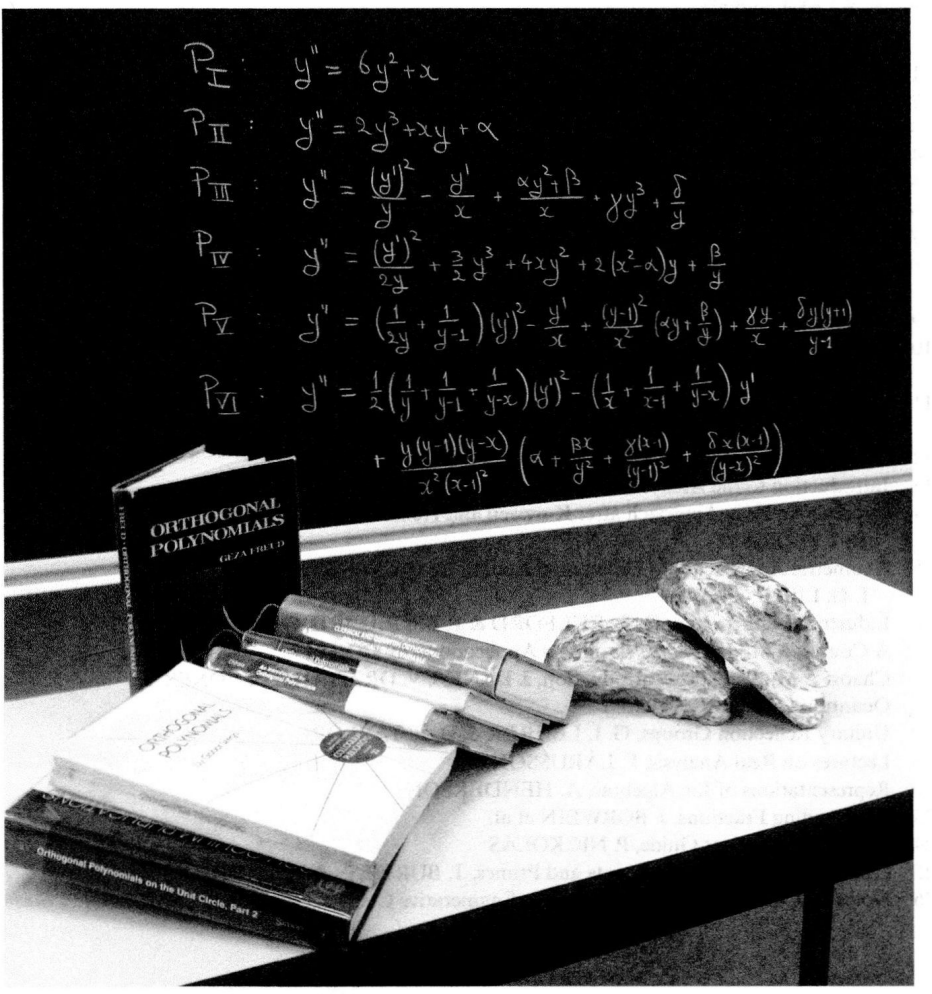
P_I : $y'' = 6y^2 + x$
P_{II} : $y'' = 2y^3 + xy + \alpha$
P_{III} : $y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}$
P_{IV} : $y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}$
P_V : $y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}$
P_{VI} : $y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y'$
$+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right)$
ORTHOGONAL POLYNOMIALS
GEZA FREUD
Orthogonal Polynomials on the Unit Circle, Part 2

Preface

These notes are intended to explain the relationship between orthogonal polynomials and Painlevé equations. They are not intended to give a systematic theory of Painlevé equations, their transformations and classification. This can be found elsewhere; we recommend in particular the classical book by Ince [88, §14.4], the more recent books [46, 68, 82, 125] and the review papers [80] and [96] on discrete and continuous Painlevé equations. Researchers in orthogonal polynomials will find the notes useful to see how semi-classical orthogonal polynomials often lead to discrete and continuous Painlevé equations. Usually only special solutions of these Painlevé equations in terms of classical special functions will be relevant. Furthermore, some integrable systems, such as the Toda lattice and related differential-difference systems, also appear in a very natural way in the theory of orthogonal polynomials. Those interested in the asymptotic behavior of orthogonal polynomials may appreciate seeing that Painlevé transcendents are used for the local analysis near critical points. Researchers in integrable systems, and in particular in Painlevé equations, may find it useful to see that a lot of explicit systems of orthogonal polynomials are described using discrete and continuous Painlevé equations. These applications in orthogonal polynomial theory often give a new viewpoint of Painlevé equations, in particular on the behavior of the special solutions of these equations.

Acknowledgments

I started writing these notes for a master course that I taught at the Universidad Carlos III de Madrid in May 2012. That course essentially consisted of Chapters 1–3. I continued these notes during my sabbatical trip to the University of Sydney in August 2016, where I started Chapters 4 and 6–7. I finally finished the notes at the African Institute for Mathematics in Cameroon in March–April

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1
Introduction

1.1 Orthogonal polynomials on the real line

Orthonormal polynomials $(p_n)_{n\in\mathbb{N}}$ on the real line are defined by the orthogonality conditions

$$\int_{\mathbb{R}} p_n(x)p_m(x)\,d\mu(x) = \delta_{m,n}, \tag{1.1}$$

where μ is a positive measure on the real line for which all the moments exist and $p_n(x) = \gamma_n x^n + \cdots$, with positive leading coefficient $\gamma_n > 0$. A family of orthonormal polynomials always satisfies a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n \geq 0, \tag{1.2}$$

with $p_{-1} = 0$ and

$$p_0 = \gamma_0 = (\mu(\mathbb{R}))^{-1/2}.$$

Comparing the leading coefficients in the recurrence relation gives

$$a_{n+1} = \frac{\gamma_n}{\gamma_{n+1}} > 0, \tag{1.3}$$

and computing the Fourier coefficients of $xp_n(x)$ in (1.2) gives

$$a_n = \int_{\mathbb{R}} xp_n(x)p_{n-1}(x)\,d\mu(x), \tag{1.4}$$

$$b_n = \int_{\mathbb{R}} xp_n^2(x)\,d\mu(x). \tag{1.5}$$

For the monic orthogonal polynomials $P_n = p_n/\gamma_n$ the recurrence relation is

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n^2 P_{n-1}(x), \tag{1.6}$$

where the b_n are as in (1.5) and the a_n^2 are the squares of (1.4).

The converse statement is also true and is known as *the spectral theorem for orthogonal polynomials*[1]: if a family of polynomials satisfies a three-term recurrence relation of the form (1.2), with $a_n > 0$ and $b_n \in \mathbb{R}$ and with initial conditions $p_0 = 1$ and $p_{-1} = 0$, then there exists a probability measure μ on the real line such that these polynomials are orthonormal polynomials satisfying (1.1). This gives rise to two important problems:

Problem 1. Suppose the measure μ is known. What can be said about the recurrence coefficients $(a_n)_{n=1,2,3,\ldots}$ and $(b_n)_{n=0,1,2,\ldots}$? This is known as the *direct problem for orthogonal polynomials.*

Problem 2. Suppose the recurrence coefficients $(a_{n+1}, b_n)_{n=0,1,2,\ldots}$ are known. What can be said about the orthogonality measure μ? This is known as the *inverse problem for orthogonal polynomials.*

A partial solution of problem 1 is that one can express the recurrence coefficients a_n^2 and b_n in terms of the moments of the measure μ. Let

$$m_n = \int_{\mathbb{R}} x^n \, d\mu(x), \qquad n \geq 0,$$

and define the Hankel determinants

$$\Delta_{n+1} = \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & \cdots & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-1} & m_{2n} \end{pmatrix}, \tag{1.7}$$

then the monic orthogonal polynomial P_n is given by

$$P_n(x) = \frac{1}{\Delta_n} \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} & m_n \\ m_1 & m_2 & m_3 & \cdots & m_n & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} & m_{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-2} & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} & x^n \end{pmatrix}. \tag{1.8}$$

From this one easily computes

$$\frac{1}{\gamma_n^2} = \int_{\mathbb{R}} P_n^2(x) \, d\mu(x) = \frac{\Delta_{n+1}}{\Delta_n},$$

[1] Also known as Favard's theorem, but the result is much older hence the attribution to Favard is not so appropriate.

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