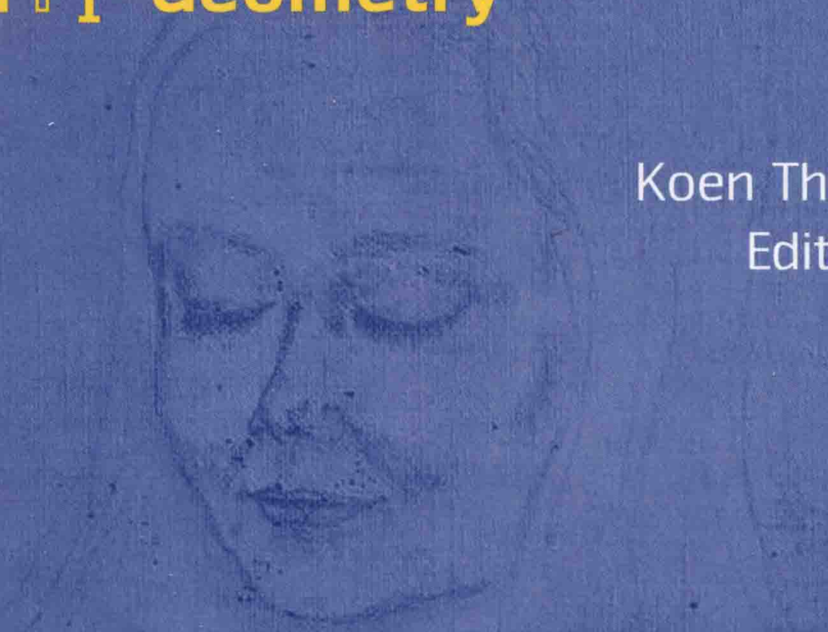
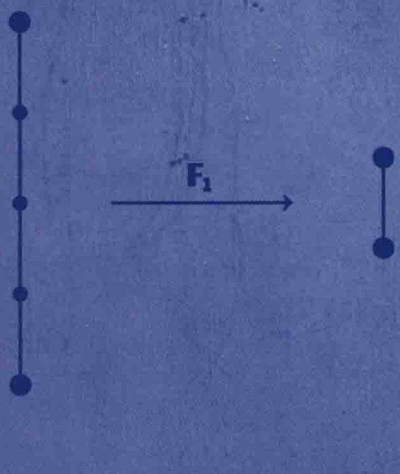
Absolute Arithmetic and $\mathbb{F}_1$-Geometry

Koen Thas
Editor





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Portrait of Innocent X

An oil on canvas (114cm × 119cm) of the Spanish painter Diego Velázquez (1599–1660) dating from about 1650, depicting a portrait of Pope Innocent X.

A casual preface

In einem unbekannten Land
vor gar nicht allzu langer Zeit
war eine Biene sehr bekannt
von der sprach alles weit und breit ...
To Maya

About ten years after Manin's lecture notes "Lectures on zeta functions and motives (according to Deninger and Kurokawa)" (dated 1995), and fifty years after Tits's influential paper "Sur les analogues algébriques des groupes semi-simples complexes" (1957), in which buildings over a "field with one element" $\mathbb{F}_1$ are described in order to see symmetric groups as Chevalley groups over this "field," the first papers got published in which scheme theories over the non-existing object $\mathbb{F}_1$ were developed. One fundamental paper is Deitmar's paper "Schemes over $\mathbb{F}_1$" in 2005 (inspired by Kato's *log schemes*); a year before, Soulé already published his $\mathbb{F}_1$-approach to varieties in "Les variétés sur le corps à un élément."

Other researchers such as Borger, Connes, Consani, Kurokawa, Lorscheid, Manin, Marcolli, Toën, and Vaquié contributed further to this rapidly emerging theory, and this stream of thoughts eventually culminated in the very recent mini-symposium "Absolute Arithmetic and $\mathbb{F}_1$-Geometry" at the 6th *European Congress of Mathematics* (Kraków, Poland) in 2012, organized by myself. The goal of the mini-symposium itself was to present the state of the art of this mysterious theory; speakers were Lieven Le Bruyn, Oliver Lorscheid, Yuri I. Manin, and myself as an extra.

Soon after, the idea grew to assemble the talks into a proceedings volume, and later Yuri Manin convinced me to see it bigger, and to aim rather for a proper monograph with chapters by various authors, so as to provide the first book on the subject of the mythical beast $\mathbb{F}_1$. And this volume is the outcome.

The book. The book is divided into four main parts:

(1) *Combinatorial Theory*—which contains one chapter (by myself);

(2) *Homological Algebra*—also containing one chapter (by Deitmar);

(3) *Algebraic Geometry*—with chapters by Borger, Le Bruyn, Lorscheid, Manin & Marcolli and myself (I refer to the table of contents for the precise titles); and

(4) *Absolute Arithmetic*—containing one chapter (by myself).

The first chapter should be seen as a combinatorial introduction on the one hand, and as a description of various combinatorial and incidence geometrical aspects of the theory on the other.

Deitmar's chapter paves a solid base for Homological Algebra of "belian categories" (certain non-additive categories like categories of modules of $\mathbb{F}_1$-algebras or $\mathbb{F}_1$-module sheaves in various $\mathbb{F}_1$-theories).

In Borger's chapter, the author extends the big and p-typical Witt vector functors from commutative rings to commutative semirings (and explains its connections with $\mathbb{F}_1$-theory).

Le Bruyn explores the origins of a new topology on the roots of unity μ_∞ introduced and studied by Kazuo Habiro in order to unify invariants of 3-dimensional homology spheres. He also seeks a meaning for the object $\mathtt{Spec}(\mathbb{Z})$ over $\mathbb{F}_1$.

Lorscheid reviews the development of $\mathbb{F}_1$-geometry from the first mentioning by Jacques Tits in 1956 until the present day. After that he explains his theory of *blueprints* in much depth (describing various connections with other scheme theories over $\mathbb{F}_1$).

Manin and Marcolli answer a question raised in the recent paper "Cyclotomy and analytic geometry over $\mathbb{F}_1$" by Manin, by showing that the genus zero moduli operad $\{\overline{M}_{0,n+1}\}$ can be endowed with natural descent data that allow it to be considered as the lift to $\mathtt{Spec}(\mathbb{Z})$ of an operad over $\mathbb{F}_1$. (They also describe a blueprint structure on $\{\overline{M}_{0,n}\}$.)

In my second chapter I first review Deitmar's theory of monoidal schemes; it is then explained how one can combinatorially study such schemes through a generalization of graph theory. In a more general setting I introduce "Υ-schemes," after which I study Grothendieck's motives in some detail in order to pass to "absolute motives" and "absolute zeta functions" (after Manin). In a last part of the chapter, I describe a marvelous connection between certain group actions on projective spaces and $\mathbb{F}_1$-theory.

Finally, I mention some aspects of "Absolute Arithmetic" in my last chapter, which may be considered as an appendix to the first three parts of the book.

Acknowledgments I want to vividly thank the authors (in alphabetical order: Jim Borger, Anton Deitmar, Lieven Le Bruyn, Oliver Lorscheid, Yuri Manin and Matilde Marcolli) for making the editorial process very pleasant. I also wish to express my deep gratitude to Manfred Karbe of the *EMS Publishing House* for helping me at various issues, and Filippo Nuccio for a splendid and energetic editing job.

Famous last words As for those readers who want to know what paintings of Velázquez and Bacon are doing in this monograph—just think of the Weyl functor.

Koen Thas
Ghent, June 2013/June 2015

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1. Introduction

We start this chapter by elaborating on ideas which were hatched from some seminal remarks made by Tits in his early paper "Sur les analogues algébriques des groupes semi-simples complexes" (1957) [20].

1.1. Projective $\mathbb{F}_1$-geometry. When considering a class of incidence geometries which are defined over finite fields—take for instance the class of finite classical buildings of a fixed rank and type (we refer to later sections for a formal explanation of these notions)—it sometimes makes sense to consider the "limit" of these geometries when the number of field elements tends to 1. As a star example, let the class of geometries be the classical projective planes $\mathbf{PG}(2,\mathbb{K})$ defined over finite fields $\mathbb{K}$. Then the number of points per line of such a plane is

$$|\mathbb{K}| + 1, \tag{1}$$

so in the limit, the "limit object" should have $1 + 1$ points incident with every line. On the other hand, we want the limit object still to be an axiomatic projective plane, so we still want it to have the following properties:

(i) any two distinct lines meet in precisely one point;

(ii) any two distinct points are incident with precisely one line (the dual of (i));

(iii) not all points are on one and the same line (to avoid degeneracy).

It is clear that such a limit projective plane "defined over $\mathbb{F}_1$" should be an ordinary triangle (as a graph). So it is nothing else than a *chamber* in the building of any thick projective plane. Note that projective planes precisely are generalized 3-gons, which are also going to be defined later.

Adopting this point of view, it is easily seen that, more generally, a projective n-space over $\mathbb{F}_1$ should be just a set X of cardinality $n+1$ endowed with the geometry of 2^X: any subset (of cardinality $0 \leq r+1 \leq n+1$) is a subspace (of dimension r). In other words, a projective n-space over $\mathbb{F}_1$ is a complete graph on $n+1$ vertices with a natural subspace structure. It is important to note that these spaces still satisfy the Veblen–Young axioms, and that they are the only such incidence geometries with thin lines.

Proposition 1.1.1 (see, e.g. Cohn [3] and Tits [20]). *Let* $n \in \mathbb{N} \cup \{-1\}$. *The combinatorial projective space* $\mathbf{PG}(n, \mathbb{F}_1) = \mathbf{PG}(n, 1)$ *is the complete graph on* $n+1$ *vertices endowed with the induced geometry of subsets, and* $\mathrm{Aut}(\mathbf{PG}(n, \mathbb{F}_1)) \cong \mathbf{PGL}_{n+1}(\mathbb{F}_1) \cong \mathbf{S}_{n+1}$.

Proof. We already have obtained the geometric part of the proposition. As for the group theoretical part, the symmetric group on $n+1$ letters clearly is the full automorphism group of $\mathbf{PG}(n, 1)$. ■

It is extremely important to note that any $\mathbf{PG}(n, \mathbb{K})$ with $\mathbb{K}$ a division ring contains (many) subgeometries isomorphic to $\mathbf{PG}(n, \mathbb{F}_1)$ as defined above; so the latter object is independent of $\mathbb{K}$, and is the *common geometric substructure of all projective spaces of a fixed given dimension*:

$$\underline{\mathscr{A}} : \{\mathbf{PG}(n, \mathbb{K}) \mid \mathbb{K} \text{ division ring }\} \longrightarrow \{\mathbf{PG}(n, \mathbb{F}_1)\}. \tag{2}$$

Further in this chapter, we will formally find the automorphism groups of $\mathbb{F}_1$-vector spaces through matrices, and these groups will perfectly agree with Proposition 1.1.1. We will also investigate other examples of limit buildings, as first described by Tits in [20]. In fact, we will look for a more general functor $\underline{\mathscr{A}}$ (called *Weyl functor* for reasons to be explained later) from a certain category of more general incidence geometries than buildings, to its subcategory of fixed objects under $\underline{\mathscr{A}}$.

Note that over $\mathbb{F}_1$,

$$\mathbf{P\Gamma L}_{n+1}(\mathbb{F}_1) \cong \mathbf{PGL}_{n+1}(\mathbb{F}_1) \cong \mathbf{PSL}_{n+1}(\mathbb{F}_1) \tag{3}$$

where $\mathbf{P\Gamma L}(\cdot)$ denotes the projective semilinear group.

1.2. Counting functions. It is easy to see the symmetric group also directly as a limit for $|\mathbb{K}| \to 1$ of linear groups $\mathbf{PG}(n, \mathbb{K})$ (with the dimension fixed). The number of elements in $\mathbf{PG}(n, \mathbb{K})$ (where $\mathbb{K} = \mathbb{F}_q$ is assumed to be finite and q is a prime power) is

$$\frac{(q^{n+1}-1)(q^{n+1}-q)\cdots(q^{n+1}-q^n)}{q-1} = (q-1)^n N(q) \tag{4}$$

for some polynomial $N(X) \in \mathbb{Z}[X]$, and we have

$$N(1) = (n+1)! = |\mathbf{S}_{n+1}|. \tag{5}$$

Now let $n, q \in \mathbb{N}$, and define $[n]_q = 1 + q + \cdots + q^{n-1}$. (For q a prime power, $[n]_q = |\mathbf{PG}(n,q)|$.) Put $[0]_q! = 1$, and define

$$[n]_q! := [1]_q[2]_q \cdots [n]_q. \tag{6}$$

Let $\mathbf{R}$ be a ring, and let x, y, q be "variables" for which $yx = qxy$. Then there are polynomials $\begin{bmatrix} n \\ k \end{bmatrix}_q$ in q with integer coefficients, such that

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}. \tag{7}$$

Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \tag{8}$$

and if q is a prime power, this is the number of $(k-1)$-dimensional subspaces of $\mathbf{PG}(n-1,q)$ $(= |\mathtt{Grass}(k,n)(\mathbb{F}_q)|)$. The next proposition again gives sense to the limit situation of q tending to 1.

Proposition 1.2.1 (see, e.g. Cohn [3]). *The number of k-dimensional linear subspaces of $\mathbf{PG}(n, \mathbb{F}_1)$, with $k \leq n \in \mathbb{N}$, equals*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_1 = \frac{n!}{(n-k)!k!} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}. \tag{9}$$

Many other enumerative formulas in Linear Algebra, Projective Geometry, etc. over finite fields $\mathbb{F}_q$ seem to keep meaningful interpretations if q tends to 1, and this phenomenon (the maintenance of various interpretations) suggests a deeper theory in characteristic one.

1.3. The Weyl functor. In this chapter, we will consider various categories $\mathtt{C}$ of combinatorial objects, and in a first stage these objects will come with certain field data (later we will also consider categories where no obvious field data are available). We will look for a functor $\underline{\mathscr{A}}$ which associates with the objects o of $\mathtt{C}$ interpretations of o over the field with one element, $\mathbb{F}_1$, keeping in mind that $\mathbb{F}_1$ does not exist, but $\underline{\mathscr{A}}(o)$ *does*. In all those categories, expressions of the form

$$\underline{\mathscr{A}}(o) \ + \ \text{field data} \tag{10}$$

make sense, in that the knowledge of $\underline{\mathscr{A}}(o)$ together with field data will single out uniquely defined objects in the $\underline{\mathscr{A}}$-fiber of o. In principle, many objects in $\mathtt{C}$ could descend to some $\underline{\mathscr{A}}(o)$, but with additional field data, we can point to a unique object. Think, for instance, about the category $\mathtt{C}$ of projective spaces over finite fields with natural morphisms; applying $\underline{\mathscr{A}}$ to $o = \mathbf{PG}(n, \mathbb{F}_q)$ yields the aforementioned geometry $\mathbf{PG}(n,1)$ which is independent of $\mathbb{F}_q$, so the $\underline{\mathscr{A}}$-fiber consists of all finite n-dimensional projective spaces. But giving the additional data of a single field yields a unique projective space coordinatized by this field. So the functor $\underline{\mathscr{A}}$ comes with a number of base extension arrows to fields, and together with these arrows, the original theories can be reconstructed from below.

Since we will consider many different categories C, we want $\underline{\mathscr{A}}$ to be defined in such a way that it commutes with various natural functors between these categories, an example of this principle being the diagram

$$\begin{array}{ccc} \{\mathbf{PG}(n,\mathbb{F}_q) \text{ for varying } n,q\} & \xrightarrow{\underline{\mathscr{A}}} & \{\mathbf{PG}(n,\mathbb{F}_1) \text{ for varying } n\} \\ \downarrow & & \downarrow \\ \{\mathbf{PGL}_{n+1}(q) \text{ for varying } n,q\} & \xrightarrow{\underline{\mathscr{A}}} & \{\mathbf{S}_{n+1} \text{ for varying } n\} \end{array} \tag{11}$$

which we have already considered.

2. BN-Pairs and the Weyl functor

Before introducing the general concepts of building and BN-pair, we study the standard example of projective spaces (from the building point of view).

2.1. Projective space. Let $\mathbf{R}$ be a division ring (= skew field), let $n \in \mathbb{N}$, and let $V = V(n,\mathbf{R}) = \mathbf{R}^n$ be the n-dimensional (left or right) vector space over $\mathbf{R}$. We define the $(n-1)$-*dimensional (left or right) projective space* $\mathbf{PG}(n,\mathbf{R})$ as being the set

$$(\mathbf{R}^n \setminus \{0\})/\sim, \tag{12}$$

where the equivalence relation "$\sim$" is defined by (left or right) proportionality, with the subspace structure being induced by that of V. (When n is not finite, similar definitions hold.) The choice of "left" or "right" does not affect the isomorphism class. If $\mathbf{R} = \mathbb{F}_q$ is the finite field with q elements (q a prime power), we also write $\mathbf{PG}(n-1,q)$ instead of $\mathbf{PG}(n-1,\mathbb{F}_q)$. Sometimes the notations $\mathbf{P}^{n-1}(\mathbf{R})$, $\mathbf{P}^{n-1}(q)$, $\mathbb{P}^{n-1}(\mathbf{R})$ and $\mathbb{P}^{n-1}(q)$ occur as well.

There is also a notion of *axiomatic projective space*, which is defined to be an *incidence geometry* (defined later in this section) which is governed by certain axioms, which are (of course) satisfied by "classical" projective spaces over division rings. A truly remarkable thing is that Veblen and Young [24] showed that if the dimension $n-1$ of such a space is at least three, it *is* isomorphic to some $\mathbf{PG}(n-1,\mathbf{R})$. And this is well known *not* to be true when the dimension is less than three.

2.2. Representing spaces as group coset geometries. Let $\mathbf{P}$ be a projective space of dimension n over some division ring $\mathbf{R}$. Consider any $\mathbf{R}$-base $\mathbf{B}$. Define a simplicial complex (in the next section to be formally defined, and called "apartment") $\mathscr{C} \equiv \mathscr{C}(\mathbf{B})$, by letting it be the set of all possible subspaces of $\mathbf{P}$ generated by subsets of $\mathbf{B}$, including the empty set. Define a (maximal) "flag" or *chamber* in $\mathscr{C}$ as a maximal chain (so of length $n+1$) of subspaces in $\mathscr{C}$. Let F be such a fixed flag.

Consider the special projective linear group $K := \mathbf{PSL}_{n+1}(\mathbf{R})$ of $\mathbf{P}$. Then note that K acts transitively on the pairs $(\mathscr{C}(\mathbf{B}'), F')$, where $\mathbf{B}'$ is any $\mathbf{R}$-base and F' is a maximal flag in $\mathscr{C}(\mathbf{B}')$. Put $B = K_{\mathscr{C}}$ and $N = K_F$; then note the following properties:

(i) $\langle B, N \rangle = K$;

(ii) put $H = B \cap N \lhd N$ and $N/H = W$; then obviously W is isomorphic to the symmetric group $\mathbf{S}_{n+1}$ on $n+1$ elements. Note that a presentation of $\mathbf{S}_{n+1}$ is:

$$\langle s_i \mid s_i^2 = \mathrm{id}, (s_i s_{i+1})^3 = \mathrm{id}, (s_i s_j)^2 = \mathrm{id} \text{ for } 1 \leq i, j \leq n+1,\ j \neq i \pm 1 \rangle. \quad (13)$$

(iii) $Bs_iBwB \subseteq BwB \cup Bs_iwB$ whenever $w \in W$ and $i \in \{1, 2, \ldots, n+1\}$;

(iv) $s_iBs_i \neq B$ for all $i \in \{1, 2, \ldots, n+1\}$.

Here, expressions such as BwB mean $B\widetilde{w}HB$, where $\widetilde{w}$ is any representant of $\widetilde{w}H = w$.

Now let $K \cong \mathbf{PSL}_{n+1}(\mathbf{R})$ be as above, and suppose that B and N are groups satisfying these properties. Define a geometry $\mathscr{B}_{(K,B,N)}$ as follows.

(B1) Elements of $\mathscr{B}_{(K,B,N)}$ are left cosets in K of the groups P_i which properly contain B and are different from K, $i = 1, \ldots, n+1$;

(B2) two elements gP_i and hP_j are incident if they intersect nontrivially.

Proposition 2.2.1. *$\mathscr{B}_{(K,B,N)}$ is isomorphic to $\mathbf{PG}(n, \mathbf{R})$.*

2.2.1. Low-dimensional cases. For dimension $n = 1$, our definition of axiomatic space doesn't make much sense. Here we rather *start* from a division ring $\mathbf{R}$, and define $\mathbf{P}$, the *projective line* over $\mathbf{R}$, as being the set $(\mathbf{R}^2 \setminus \{0\})/\sim$, where $\sim$ is defined by (left) proportionality. So we can write

$$\mathbf{P} = \{(0,1)\} \cup \{(1,\ell) \mid \ell \in \mathbf{R}\}. \quad (14)$$

Now $\mathbf{PSL}_2(\mathbf{R})$ acts naturally on $\mathbf{P}$; in fact, we have defined the projective line as a permutation group equipped with the natural doubly transitive action of $\mathbf{PSL}_2(\mathbf{R})$. Defining a geometry as we did for higher-rank projective spaces, through the "(B, N)-pair structure" of $\mathbf{PSL}_2(\mathbf{R})$, one obtains the same notion of projective line.