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Peter Hagedorn
Gottfried Spelsberg-Korspeter *Editors*

Active and Passive Vibration Control of Structures

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Peter Hagedorn · Gottfried Spelsberg-Korspeter
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Peter Hagedorn
TU Darmstadt, Darmstadt, Germany

Gottfried Spelsberg-Korspeter
TU Darmstadt, Darmstadt, Germany

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PREFACE

Active and passive vibration control of structures form a topic of very actual interest in many different fields of engineering, for example in the automotive industry, in aerospace engineering (e.g. in large telescopes) and also in civil engineering. The material presented in this book tries to fill gaps between structural mechanics, vibrations and modern control theory. It also establishes links between the different applications in structural control. In vibration control it is still common practice to a large extent to regard the design of mechanical structures and their damping and control as different topics, which often are treated sequentially, with the design of the structure to be carried out first, followed later by designing the active and passive vibration control. Similarly, in the standard curricula at most universities, the mechanical modelling and the design of structures as well as their control are treated as separate subjects. This book intends to fill some of the gaps between these different issues. The intention is to give a solid foundation of the mechanical modelling and the vibration control for discrete and continuous structures, with an emphasis on the interfaces of the different disciplines.

This book was written accompanying the CISM Course no. 418 entitled 'Active and Passive Vibration Control of Structures' *held in Udine from May 27 to 31, 2013. Therefore it is directed to young researchers, to doctoral students and also to engineers working in fields related to structures, vibrations and control.*

A thorough introduction into the relevant theory both of the mechanical modelling as well as of the vibration control theory are presented and the most important design goals are discussed. Various strategies for modelling complex mechanical structures are given and an introduction to active, passive and semi-active strategies for vibration control are discussed. In a number of examples from different areas it is shown that a comprehensive approach, in which both the mechanical design problem and the development of suitable controls are considered simultaneously, can present substantial advantages.

The organization of the book is as follows.

Chapter I, by PETER HAGEDORN *(Technische Universität Darmstadt, Germany), treats equations of motion for discrete and con-*

tinuous mechanical systems laying the foundation for the creation of control models.

Chapter II, by Gottfried Spelsberg-Korspeter *(Technische Universität Darmstadt, Germany), gives an introduction to variational principles in mechanics and control relating to mechanical modelling and the development of control strategies.*

Chapter III, by Andre Preumont, David Alaluf *and* Renaud Bastaits *(Université Libre de Bruxelles, Belgium), treat hybrid mass dampers for mitigating the dynamic response of buildings.*

Chapter IV, by Andre Preumont *and* Bilal Mokrani *(Université Libre de Bruxelles, Belgium), discuss the theory of electromagnetic and piezoelectric transducers and presents some applications in structural control.*

Chapter V, by Robert Skelton *(University of California San Diego, United States), focuses on structure and design of control systems with an emphasis on the advantages of using matrix inequalities.*

Chapter VI, by Adnan Akay *(Bilkent University, Turkey) and* Antonio Carcaterra *(University of Rome, Italy), address the physics and modelling of structural damping which is extremely important in almost every structural control problem.*

Chapter VII, by Rainer Nordmann *(Technische Universität Darmstadt, Germany), deals with active magnetic bearings, which can be used for control purposes in rotating machinery.*

Summarizing, the material presented in this book will offer a unified view on active and passive control and the mechanical modelling of structures presented from the point of view of experienced researchers with quite different perspectives.

The authors want to express their gratitude for the support of the CISM organization, in particular to Professor Friedrich Pfeiffer for chairing the course in Udine, and to Carla Toros for her tremendous support in organizing it.

The authors sincerely thank Manuel Eckstein, who carried the main burden of editing the manuscripts and coordinating the different chapters, as well to Eduard Heffel, Matthias Heymanns, Henning Spiegelberg and Andreas Wagner.

Peter Hagedorn
Gottfried Spelsberg-Korspeter

CONTENTS

Mechanical Systems: Equations of Motion and Stability

Peter Hagedorn
TU Darmstadt, Darmstadt, Germany

Abstract The Chapter 'Mechanical Systems: Equations of Motion and Stability' corresponds to the material presented in five lectures given at the CISM Course no. 418. The first parts deal with the form of the equations of motion of mechanical systems, in particular the linearized equations and the influence and importance of the different terms (inertia terms, damping, gyroscopic terms, restoring terms and circulatory terms as well as with their physical origin). This is done both for discrete systems, and the corresponding material is part of the recent book *Hagedorn & Hochlenert, Technische Schwingungslehre, Verlag Harri Deutsch, Frankfurt, 2012*, as well as for continuous systems, the material being adapted from *Hagedorn & DasGupta, Vibrations and Waves in Continuous Mechanical Systems, Wiley, Chichester, 2007.* Almost all the material is presented in typical elementary vibration courses, but here certain aspects will be highlighted, which are not always stressed in basic vibration courses. The third part deals with LIAPOUNOV stability, the material is from the author's earlier book *Hagedorn, Non-Linear Oscillations, 2nd edition, Oxford Science Publications, 1988.* The material of these five lectures is used in the other lectures of the course.

The author prepared most of the material in 2012 and 2013, while staying at the University of Canterbury in Christchurch, New Zealand. The author thanks the Department of Mechanical Engineering of the UC for providing the infrastructure and assistance.

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1 Equations of Motion of Discrete Mechanical Systems

In this lecture we will shortly recapitulate the form of the equations of motion of discrete mechanical systems (which may of course be an approximation of continuous systems). We will highlight certain aspects which, although elementary, are not always stressed in basic vibration courses.

For a holonomic system of n degrees of freedom and generalized coordinates

$$\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^{\mathsf{T}} \tag{1}$$

the equations of motion can be obtained from the LAGRANGE equations of the second type, based on the Lagrangian

$$L = T - U, \tag{2}$$

where T is the kinetic energy and U the potential energy function (which we assume may also depend on the time t). LAGRANGE's equations then read

$$\frac{\partial}{\partial t}\frac{\partial L}{\dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s, \tag{3}$$

where the Q_s are the generalized forces not represented by the potential U. For a large class of systems these equations can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t). \tag{4}$$

The term $\boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ contains for example the damping and other nonconservative terms, as well as for example control forces. The linearized equations (linearized about an equilibrium of the unforced autonomous system) can then be written as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + (\boldsymbol{D} + \boldsymbol{G})\dot{\boldsymbol{q}} + (\boldsymbol{K} + \boldsymbol{N})\boldsymbol{q} = \boldsymbol{f}(t). \tag{5}$$

This linearized form of the equations of motion is usually employed to develop appropriate active or passive vibration control. The control strategies based on these linear models may then later be tested for the nonlinear model.

In many cases, setting up the equations of motion using LAGRANGE's equations is not a practical approach and other methods may be more efficient. The form of the equations will however be the same as above. Unless

stated otherwise, we will assume the following properties for the matrices:

Mass matrix:	$\boldsymbol{M} = \boldsymbol{M}^{\mathsf{T}}$,	$\boldsymbol{M} > 0$	(symmetric, positive definite)
Damping matrix:	$\boldsymbol{D} = \boldsymbol{D}^{\mathsf{T}}$,	$\boldsymbol{D} \geq 0$	(symmetric, positive semidef.)
Stiffness matrix:	$\boldsymbol{K} = \boldsymbol{K}^{\mathsf{T}}$,	$\boldsymbol{K} \geq 0$	(symmetric, positive semidef.)
Gyroscopic matrix:	$\boldsymbol{G} = -\boldsymbol{G}^{\mathsf{T}}$,		(skew symmetric)
Circulatory matrix:	$\boldsymbol{N} = -\boldsymbol{N}^{\mathsf{T}}$,		(skew symmetric)

In this lecture and in the next one, we will discuss in some more detail the significance of the different matrices for the behavior of the mechanical systems.

We will first consider the free vibrations, i.e. the case $\boldsymbol{f}(t) = 0$:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + (\boldsymbol{D} + \boldsymbol{G})\dot{\boldsymbol{q}} + (\boldsymbol{K} + \boldsymbol{N})\boldsymbol{q} = \boldsymbol{0}. \tag{6}$$

1.1 The Eigenvalue Problem

Since (5) is a system of ode's with constant coefficients, the exponential *ansatz*

$$\boldsymbol{q}(t) = \boldsymbol{r}e^{\lambda t} \tag{7}$$

is successful, leading to

$$\Big[\lambda^2\boldsymbol{M} + \lambda(\boldsymbol{D} + \boldsymbol{G}) + \boldsymbol{K} + \boldsymbol{N}\Big]\boldsymbol{r}e^{\lambda t} = \boldsymbol{0}. \tag{8}$$

In order for (8) to be valid for all times, the condition

$$\Big[\lambda^2\boldsymbol{M} + \lambda(\boldsymbol{D} + \boldsymbol{G}) + \boldsymbol{K} + \boldsymbol{N}\Big]\boldsymbol{r} = \boldsymbol{0} \tag{9}$$

must be fulfilled. Equation (9) is the eigenvalue problem, and the sought values of λ and $\boldsymbol{r}$ are respectively the *eigenvalues* and the *eigenvectors*.

The eigenvalue problem is a linear homogeneous algebraic system in the unknown vector $\boldsymbol{r}$. A necessary condition for the existence of non-trivial solutions in $\boldsymbol{r}$, is that the determinant of the coefficient matrix vanishes, and this leads to the characteristic equation

$$\det\Big(\lambda^2\boldsymbol{M} + \lambda(\boldsymbol{D} + \boldsymbol{G}) + \boldsymbol{K} + \boldsymbol{N}\Big) = 0. \tag{10}$$

The left-hand side is a polynomial of degree $2n$ in λ and can be written as

$$a_{2n}\lambda^{2n} + \ldots + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{11}$$

The $2n$ solutions λ_i $(i = 1, 2, \ldots, 2n)$ are real or appear in complex conjugate pairs, since all the coefficients a_k are real. For each eigenvalue λ_i the corresponding nontrivial eigenvector $\boldsymbol{r}_i$ can then be calculated from

$$\Big[\lambda_i^2\boldsymbol{M} + \lambda_i(\boldsymbol{D} + \boldsymbol{G}) + \boldsymbol{K} + \boldsymbol{N}\Big]\boldsymbol{r}_i = \boldsymbol{0}, \qquad i = 1, \ldots, 2n. \tag{12}$$

We will not deal with the case of multiple eigenvalues with multiplicity larger than the number of independent eigenvectors; the case of 'semi-simple' multiple eigenvalues will however play an important role in a later lecture. Obviously the eigenvectors resulting from (12) can at most be determined up to a multiplicative (complex) factor, since we found the eigenvalues by setting the determinant of the coefficient matrix equal to zero and the rows and columns of the coefficient matrix are therefore linearly dependent.

Only in very particular cases, matrix eigenvalue problems can be solved analytically. As a rule, eigenvalues and eigenvectors will be numerically calculated using appropriate numerical tools. For example MATLAB immediately calculates the eigenvalues and eigenvectors with the command `polyeig(K+N,D+G,M)`, if the corresponding numerical values are attributed to the matrices `M`, `D`, `G`, `K` and `N`.

Normalization of Eigenvectors It is often important to normalize the eigenvectors, if only for example to compare numerical or experimental results. From (12) it is obvious that a complex eigenvalue will in general imply a complex eigenvector $\boldsymbol{r}_i$, and that a real eigenvector can be found for each real eigenvalue. Real eigenvectors can be normalized e.g. according to

$$\boldsymbol{r}_i^{\mathsf{T}}\boldsymbol{r}_i = 1 \qquad \text{or} \qquad \boldsymbol{r}_i^{\mathsf{T}}\boldsymbol{M}\boldsymbol{r}_i = 1 \tag{13a}$$

with respect to their magnitude or with respect to the mass matrix. Let $\underline{\boldsymbol{r}}_i$ be a non-normalized eigenvector, then the corresponding normalized eigenvector $\boldsymbol{r}_i$ is

$$\boldsymbol{r}_i = \frac{\underline{\boldsymbol{r}}_i}{\sqrt{\underline{\boldsymbol{r}}_i^{\mathsf{T}}\underline{\boldsymbol{r}}_i}} \qquad \text{or} \qquad \boldsymbol{r}_i = \frac{\underline{\boldsymbol{r}}_i}{\sqrt{\underline{\boldsymbol{r}}_i^{\mathsf{T}}\boldsymbol{M}\underline{\boldsymbol{r}}_i}}. \tag{13b}$$

This eigenvector is still not uniquely determined, as is obvious that the real eigenvector $\tilde{\boldsymbol{r}}_i = -\boldsymbol{r}_i$ also fulfills the normalization. This is more involved for complex eigenvectors, whose absolute value can be normalized in analogy to (13a) with

$$\boldsymbol{r}_i^*\boldsymbol{r}_i = 1 \qquad \text{or} \qquad \boldsymbol{r}_i^*\boldsymbol{M}\boldsymbol{r}_i = 1. \tag{13c}$$

Here $\boldsymbol{r}_i^*$ is the complex conjugate transposed vector to $\boldsymbol{r}_i$. Since the normalization of complex eigenvalues is less commonly discussed in the literature, we will describe it in more detail. The normalization (13c) is here also fulfilled by the complex eigenvector $\tilde{\boldsymbol{r}}_i = e^{j\beta_i}\boldsymbol{r}_i$ for an arbitrary β_i, i.e. each component of $\boldsymbol{r}_i$ can still be rotated in the complex plane by an equal but arbitrary angle β_i. The components of the real eigenvectors can in the same fashion be rotated by the angle π, remaining real, or by an arbitrary angle, so that the eigenvectors become complex.

In order to obtain uniquely determined eigenvectors, so that the results of different computations can be compared with each other, it is convenient to fix the angle β_i conveniently. This can be done by choosing β_i so that a complex eigenvector is reduced to a real form, if this is possible. As an example, consider the eigenvector

$$\boldsymbol{r} = \boldsymbol{a} + j\boldsymbol{b} \tag{14}$$

with real part $\boldsymbol{a}$ and imaginary part $\boldsymbol{b}$, as well as its representation rotated by an angle β

$$\begin{aligned}\tilde{\boldsymbol{r}} &= e^{j\beta}(\boldsymbol{a} + j\boldsymbol{b}) = (\boldsymbol{a}\cos\beta - \boldsymbol{b}\sin\beta) + j(\boldsymbol{a}\sin\beta + \boldsymbol{b}\cos\beta) \\ &= \tilde{\boldsymbol{a}} + j\tilde{\boldsymbol{b}}.\end{aligned} \tag{15}$$

The angle β will now be chosen in such a way that the real part of $\tilde{\boldsymbol{r}}$ becomes 'as large as possible'. We formulate this requirement by

$$\begin{aligned}\max_\beta \tilde{\boldsymbol{a}}^\mathsf{T}\tilde{\boldsymbol{a}} &= \max_\beta (\boldsymbol{a}\cos\beta - \boldsymbol{b}\sin\beta)^\mathsf{T}(\boldsymbol{a}\cos\beta - \boldsymbol{b}\sin\beta) \\ &= \max_\beta \ \boldsymbol{a}^\mathsf{T}\boldsymbol{a}\,\cos^2\beta - 2\boldsymbol{a}^\mathsf{T}\boldsymbol{b}\,\cos\beta\sin\beta + \boldsymbol{b}^\mathsf{T}\boldsymbol{b}\,\sin^2\beta \\ &= \max_\beta \begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix}\begin{bmatrix}\boldsymbol{a}^\mathsf{T}\boldsymbol{a} & -\boldsymbol{a}^\mathsf{T}\boldsymbol{b} \\ -\boldsymbol{a}^\mathsf{T}\boldsymbol{b} & \boldsymbol{b}^\mathsf{T}\boldsymbol{b}\end{bmatrix}\begin{bmatrix}\cos\beta \\ \sin\beta\end{bmatrix},\end{aligned} \tag{16}$$

i.e. by the maximization of a positive definite quadratic form with the corresponding eigenvalue problem in μ

$$\begin{bmatrix}\boldsymbol{a}^\mathsf{T}\boldsymbol{a} - \mu & -\boldsymbol{a}^\mathsf{T}\boldsymbol{b} \\ -\boldsymbol{a}^\mathsf{T}\boldsymbol{b} & \boldsymbol{b}^\mathsf{T}\boldsymbol{b} - \mu\end{bmatrix}\begin{bmatrix}\cos\beta \\ \sin\beta\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}. \tag{17}$$

The two eigenvalues for (17) are

$$\mu_{1,2} = \frac{\boldsymbol{a}^\mathsf{T}\boldsymbol{a} + \boldsymbol{b}^\mathsf{T}\boldsymbol{b}}{2} \pm \sqrt{\frac{(\boldsymbol{a}^\mathsf{T}\boldsymbol{a} - \boldsymbol{b}^\mathsf{T}\boldsymbol{b})^2}{4} - (\boldsymbol{a}^\mathsf{T}\boldsymbol{b})^2}\,. \tag{18}$$

Substituting the larger of the two eigenvalues μ_2 in (17) leads to the two equations

$$\left(\boldsymbol{b}^\top\boldsymbol{b} - \boldsymbol{a}^\top\boldsymbol{a} + \sqrt{(\boldsymbol{b}^\top\boldsymbol{b} - \boldsymbol{a}^\top\boldsymbol{a})^2 - 4(\boldsymbol{a}^\top\boldsymbol{b})^2}\right)\cos\beta + 2\boldsymbol{a}^\top\boldsymbol{b}\sin\beta = 0, \quad (19a)$$

$$2\boldsymbol{a}^\top\boldsymbol{b}\cos\beta + \left(\boldsymbol{a}^\top\boldsymbol{a} - \boldsymbol{b}^\top\boldsymbol{b} + \sqrt{(\boldsymbol{a}^\top\boldsymbol{a} - \boldsymbol{b}^\top\boldsymbol{b})^2 - 4(\boldsymbol{a}^\top\boldsymbol{b})^2}\right)\sin\beta = 0 \quad (19b)$$

for the determination of $\cos\beta$ and $\sin\beta$. The two equations are of course linearly dependent, but one should be careful in selecting one of the equations. For $\boldsymbol{a}^\top\boldsymbol{b}=0$ and $\boldsymbol{a}^\top\boldsymbol{a}=\boldsymbol{b}^\top\boldsymbol{b}$ both equations are identically fulfilled, independently of the value of β. In this case we choose $\beta=0$. For $\boldsymbol{a}^\top\boldsymbol{b}=0$ and $\boldsymbol{a}^\top\boldsymbol{a}>\boldsymbol{b}^\top\boldsymbol{b}$ both coefficients of the first equation vanish and the second equation gives $\sin\beta=0$. For $\boldsymbol{a}^\top\boldsymbol{b}=0$ and $\boldsymbol{a}^\top\boldsymbol{a}<\boldsymbol{b}^\top\boldsymbol{b}$ the first equation gives $\cos\beta=0$, and both coefficients of the second equation vanish. For $\boldsymbol{a}^\top\boldsymbol{b}\neq 0$ either the first or the second equation can be used. Summarizing, one has

$$\tan\beta = \begin{cases} 0 & \text{for} \quad \boldsymbol{a}^\top\boldsymbol{b} = 0,\ \boldsymbol{a}^\top\boldsymbol{a} \geq \boldsymbol{b}^\top\boldsymbol{b} \\ \pm\infty & \text{for} \quad \boldsymbol{a}^\top\boldsymbol{b} = 0,\ \boldsymbol{a}^\top\boldsymbol{a} < \boldsymbol{b}^\top\boldsymbol{b} \\ \dfrac{\boldsymbol{a}^\top\boldsymbol{a} - \boldsymbol{b}^\top\boldsymbol{b}}{2\boldsymbol{a}^\top\boldsymbol{b}} - \sqrt{\dfrac{(\boldsymbol{a}^\top\boldsymbol{a} - \boldsymbol{b}^\top\boldsymbol{b})^2}{4(\boldsymbol{a}^\top\boldsymbol{b})^2} - 1} & \text{for} \quad \boldsymbol{a}^\top\boldsymbol{b} \neq 0 \end{cases} . \quad (20)$$

In case one desires to maximize the real part with respect to the mass matrix, according to $\max \tilde{\boldsymbol{a}}^\top\boldsymbol{M}\tilde{\boldsymbol{a}}$, the matrix $\boldsymbol{M}$ is introduced in the scalar products. A normalization of the eigenvectors with β according to (20) subject to (13c), will therefore always result in a real representation of the eigenvector, if this is at all possible. In order to make the normalization completely unique, since (20) still permits a factor -1, one may for example prescribe that the first non vanishing component of $\boldsymbol{a}$ be positive.

For many applications, the normalization according to (13) is completely sufficient. Yet, for the comparison of different numerical results the determination of β will in some cases be necessary and a completely unique normalization of the eigenvectors may be in order.

General Solution for Free Vibrations For distinct eigenvalues ($\lambda_i \neq \lambda_k$) (12) gives $2n$ linear independent solutions of the form $\boldsymbol{r}_i\, e^{\lambda_i t}$. With complex integration constants, adding the individual solutions leads to the general

solution of (6) in the form

$$\boldsymbol{q}(t) = \sum_{i=1}^{2n} \underset{\sim}{K}_i \, \boldsymbol{r}_i \, e^{\lambda_i t}. \tag{21}$$

The integration constants $\underset{\sim}{K}_i$ $(i = 1, \ldots, 2n)$ can be determined from the initial conditions

$$\boldsymbol{q}(0) = \boldsymbol{q}_0, \qquad \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_0. \tag{22}$$

For real initial conditions, $\boldsymbol{q}(t)$ will then also be real.

It may however be convenient to directly write (21) in real form. In doing this we assume first that the eigenvalues are all complex. From (12) one recognizes that for each pair of complex conjugate eigenvalues there is a pair of complex conjugate eigenvectors, so that

$$\lambda_{i,n+i} = -\delta_i \pm j\omega_{\mathrm{d}i}, \qquad \boldsymbol{r}_{i,n+i} = \boldsymbol{a}_i \pm j\boldsymbol{b}_i \tag{23}$$

holds. Next, we choose the integration constants in complex conjugate pairs $\underset{\sim}{K}_i = \underset{\sim}{K}^*_{n+i}$ and write them as

$$\underset{\sim}{K}_i = \frac{1}{2} K_i \, e^{j\gamma_i}, \qquad \underset{\sim}{K}_{n+i} = \frac{1}{2} K_i \, e^{-j\gamma_i} \tag{24}$$

with the new integration constants K_i and γ_i. With this substitution, (21) assumes the real form

$$\begin{aligned}
\boldsymbol{q}(t) &= \sum_{i=1}^{n} \frac{1}{2} K_i \, (\boldsymbol{a}_i + j\boldsymbol{b}_i) \, e^{-\delta_i t + j(\omega_{\mathrm{d}i} t + \gamma_i)} + \frac{1}{2} K_i \, (\boldsymbol{a}_i - j\boldsymbol{b}_i) \, e^{-\delta_i t - j(\omega_{\mathrm{d}i} t + \gamma_i)} \\
&= \sum_{i=1}^{n} K_i \, e^{-\delta_i t} \Big[\boldsymbol{a}_i \Big(e^{j(\omega_{\mathrm{d}i} t + \gamma_i)} + e^{-j(\omega_{\mathrm{d}i} t + \gamma_i)} \Big) \\
&\qquad\qquad + j\boldsymbol{b}_i \Big(e^{j(\omega_{\mathrm{d}i} t + \gamma_i)} - e^{-j(\omega_{\mathrm{d}i} t + \gamma_i)} \Big) \Big] \\
&= \sum_{i=1}^{n} K_i \, e^{-\delta_i t} \Big[\boldsymbol{a}_i \cos(\omega_{\mathrm{d}i} t + \gamma_i) - \boldsymbol{b}_i \sin(\omega_{\mathrm{d}i} t + \gamma_i) \Big].
\end{aligned} \tag{25a}$$

If the system does not have $2n$ complex eigenvalues, but only $2s$ complex eigenvalues $\lambda_{i,s+i}$ $(i = 1, \ldots, s)$ and $2(n-s)$ real eigenvalues λ_i $(i = 2s+1, \ldots, 2n)$ with the corresponding real eigenvector, one has

$$\boldsymbol{q}(t) = \sum_{i=1}^{s} K_i \, e^{-\delta_i t} \Big[\boldsymbol{a}_i \cos(\omega_{\mathrm{d}i} t + \gamma_i) - \boldsymbol{b}_i \sin(\omega_{\mathrm{d}i} t + \gamma_i) \Big] + \sum_{i=2s+1}^{2n} K_i \, \boldsymbol{r}_i \, e^{\lambda_i t}. \tag{25b}$$

The behavior of the different particular solutions corresponding to the different eigenvalues strongly depends in particular on the real parts of the eigenvalues, as is well known. They determine the stability of the solutions and of the system. An important aspect of (25) is that the eigensolutions in general (when all the γ are different from zero) are such that their different coordinates vanish at different times. This is quite different in the particular case which we examine next.

1.2 *M*-*K*-Systems

An important particular case is that in which the matrices $\boldsymbol{D}$, $\boldsymbol{G}$ and $\boldsymbol{N}$ vanish. The equations of motion are then of the form

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0}. \tag{26}$$

We will recapitulate their main properties and then later examine the way in which the other terms in the equations of motion influence the solutions.

Properties of the Eigenvalues For $\boldsymbol{M}$-$\boldsymbol{K}$-systems the characteristic equation (10) simplifies to

$$\det(\lambda^2\boldsymbol{M} + \boldsymbol{K}) = 0. \tag{27}$$

This is a polynomial of degree n in λ^2. The roots λ_i^2 $(i = 1, \ldots, n)$ not necessarily are single. If they are, then for each λ_i^2 there is exactly one nontrivial possibly complex eigenvector $\boldsymbol{r}_i$, satisfying

$$(\lambda_i^2\boldsymbol{M} + \boldsymbol{K})\boldsymbol{r}_i = \boldsymbol{0}. \tag{28}$$

Multiplying (28) from the left with the vector $\boldsymbol{r}_i^*$ leads to

$$\lambda_i^2\, \boldsymbol{r}_i^*\boldsymbol{M}\boldsymbol{r}_i + \boldsymbol{r}_i^*\boldsymbol{K}\boldsymbol{r}_i = 0. \tag{29}$$

In doing this we have in a way 'projected' the eigenvalue problem (28) on $\boldsymbol{r}_i^*$, which of course leads to a loss of information. It would not be correct to assume that all the solutions of (29) also fulfill (28). The inverse is however true.

Since $\boldsymbol{M}$ and $\boldsymbol{K}$ were assumed symmetric, the expressions $\boldsymbol{r}_i^*\boldsymbol{M}\boldsymbol{r}_i$ and $\boldsymbol{r}_i^*\boldsymbol{K}\boldsymbol{r}_i$ are real for arbitrary vectors $\boldsymbol{r}_i$. For a positive definite matrix one has $\boldsymbol{r}_i^*\boldsymbol{M}\boldsymbol{r}_i > 0$ and (29) can be transformed into

$$\lambda_i^2 = -\frac{\boldsymbol{r}_i^*\boldsymbol{K}\boldsymbol{r}_i}{\boldsymbol{r}_i^*\boldsymbol{M}\boldsymbol{r}_i}. \tag{30}$$

If also $\boldsymbol{K}$ is positive definite, we can conclude that λ^2 is a negative real number. Since we obtained this result from the 'projected' equation (29), at this point we only can be sure that one of the two solutions of (30) will also fulfill (28). We do however know that the eigenvalues appear in complex conjugate pairs

$$\lambda_i = \lambda_{n+i}^* , \qquad i = 1, \ldots, n, \tag{31}$$

so that in fact both solutions of (30) fulfill (28). The $2n$ eigenvalus

$$\lambda_{i,n+i} = \pm j \sqrt{\frac{\boldsymbol{r}_i^* \boldsymbol{K} \boldsymbol{r}_i}{\boldsymbol{r}_i^* \boldsymbol{M} \boldsymbol{r}_i}} , \qquad i = 1, \ldots, n \tag{32}$$

of any $\boldsymbol{M}$-$\boldsymbol{K}$-system (26) are therefore purely imaginary (if $\boldsymbol{K}$ is positive definite). This according to (28) implies that the eigenvectors can be chosen real, where $\boldsymbol{r}_i$ is the eigenvector associated to λ_i and λ_i^*. If the stiffness matrix is not positive definite but only positive semidefinite ($\boldsymbol{r}_i^\mathsf{T} \boldsymbol{K} \boldsymbol{r}_i \geq 0$), then there are pairs of zero eigenvaluse $\lambda_i = \lambda_{n+i} = 0$; with positive definite stiffness matrix ($\boldsymbol{r}_i^\mathsf{T} \boldsymbol{K} \boldsymbol{r}_i > 0$) all the eigenvalues are complex conjugate in strict sense.

Orthogonality Relations of Eigenvectors If $(\lambda_i, \boldsymbol{r}_i)$ and $(\lambda_k, \boldsymbol{r}_k)$ are two eigenpairs, i.e. pairs of eigenvalues with the corresponding eigenvectors, then (29) impies

$$\lambda_i^2 \boldsymbol{M} \boldsymbol{r}_i + \boldsymbol{K} \boldsymbol{r}_i = \boldsymbol{0}, \tag{33a}$$

$$\lambda_k^2 \boldsymbol{M} \boldsymbol{r}_k + \boldsymbol{K} \boldsymbol{r}_k = \boldsymbol{0}. \tag{33b}$$

Multiplying (33a), respectively (33b) from the left with $\boldsymbol{r}_k^\mathsf{T}$, respectively with $\boldsymbol{r}_i^\mathsf{T}$, leads to

$$\lambda_i^2 \, \boldsymbol{r}_k^\mathsf{T} \boldsymbol{M} \boldsymbol{r}_i + \boldsymbol{r}_k^\mathsf{T} \boldsymbol{K} \boldsymbol{r}_i = 0, \tag{34a}$$

$$\lambda_k^2 \, \boldsymbol{r}_i^\mathsf{T} \boldsymbol{M} \boldsymbol{r}_k + \boldsymbol{r}_i^\mathsf{T} \boldsymbol{K} \boldsymbol{r}_k = 0. \tag{34b}$$

Due to the symmetry of $\boldsymbol{M}$ and $\boldsymbol{K}$ we have $\boldsymbol{r}_k^\mathsf{T} \boldsymbol{M} \boldsymbol{r}_i = \boldsymbol{r}_i^\mathsf{T} \boldsymbol{M} \boldsymbol{r}_k$ and $\boldsymbol{r}_k^\mathsf{T} \boldsymbol{K} \boldsymbol{r}_i = \boldsymbol{r}_i^\mathsf{T} \boldsymbol{K} \boldsymbol{r}_k$, so that the the difference of (34a) and (34b) gives

$$(\lambda_i^2 - \lambda_k^2) \, \boldsymbol{r}_i^\mathsf{T} \boldsymbol{M} \boldsymbol{r}_k = 0. \tag{35}$$

This finally leads to

$$\boldsymbol{r}_i^\mathsf{T} \boldsymbol{M} \boldsymbol{r}_k = 0 \qquad \text{for} \qquad \lambda_i^2 \neq \lambda_k^2. \tag{36a}$$

The eigenvectors associated to different eigenvalues ($\lambda_i^2 \neq \lambda_k^2$) are orthogonal with respect to the mass matrix. Similary one also has

$$\boldsymbol{r}_i^\top \boldsymbol{K} \boldsymbol{r}_k = 0 \qquad \text{for} \qquad \lambda_i^2 \neq \lambda_k^2, \tag{36b}$$

i.e. the eigenvectors are also orthogonal with respect to the stiffness matrix. In general the eigenvectors will however not be orthogonal in the usual sense, i.e. with respect to the identity matrix!

For multiple eigenvalues, i.e. for $\lambda_i^2 = \lambda_k^2$, (35) does not imply the orthogonality of the corresponding eigenvectors. However, if the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are symmetric, as we always assume according to our definition (5), it can be shown that to each pair of eigenvalues of multiplicity m, exactly m linearly independent eigenvectors exist, which are all orthogonal to the remaining $n-m$ eigenvectors. Each linear combination of these eigenvectors is again an eigenvector λ_i^2. Using for example the GRAM-SCHMIDT orthogonalisation procedure, one can now construct an orthogonal basis (orthogonal with respect to $\boldsymbol{M}$) forming a basis of the subspace of $\mathbb{R}^n$ spanned by the m eigenvectors. Doing that for all eigenvectors pertaining to multiple eigenvalues, again results in n eigenvectors mutually orthogonal with respect to $\boldsymbol{M}$ and $\boldsymbol{K}$.

The orthogonality of the eigenvectors has the consequence that $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ is a basis of $\mathbb{R}^n$. In fact, consider

$$e_1 \boldsymbol{r}_1 + e_2 \boldsymbol{r}_2 + \ldots + e_n \boldsymbol{r}_n = \boldsymbol{0}, \tag{37}$$

i.e. a linear combination of the eigenvectors giving the zero vector, than, multiplication with $\boldsymbol{r}_i^\top \boldsymbol{M}$ from the left leads to

$$e_i \, \boldsymbol{r}_i^\top \boldsymbol{M} \boldsymbol{r}_i = 0, \tag{38}$$

i.e.

$$e_i = 0, \qquad i = 1, \ldots, n \tag{39}$$

and this is the condition for linear independence. Since the eigenvectors form a basis of $\mathbb{R}^n$, any vector $\boldsymbol{a}$ of $\mathbb{R}^n$ has a unique representation in form of

$$\boldsymbol{a} = a_1 \boldsymbol{r}_1 + a_2 \boldsymbol{r}_2 + \ldots + a_n \boldsymbol{r}_n. \tag{40}$$

Multiplying from the left with $\boldsymbol{r}_i^\top \boldsymbol{M}$ and using orthogonality of the eigenvectors leads to

$$\boldsymbol{r}_i^\top \boldsymbol{M} \boldsymbol{a} = a_i \, \boldsymbol{r}_i^\top \boldsymbol{M} \boldsymbol{r}_i, \tag{41}$$

so that the coefficients a_i are obviously given by

$$a_i = \frac{\boldsymbol{r}_i^\mathsf{T}\boldsymbol{M}\boldsymbol{a}}{\boldsymbol{r}_i^\mathsf{T}\boldsymbol{M}\boldsymbol{r}_i} = \frac{\boldsymbol{r}_i^\mathsf{T}\boldsymbol{K}\boldsymbol{a}}{\boldsymbol{r}_i^\mathsf{T}\boldsymbol{K}\boldsymbol{r}_i}, \qquad i = 1, \ldots, n. \tag{42}$$

Modal Coordinates The eigenvectors can be assembled column wise in the so called *modal matrix*

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_n \end{bmatrix} \tag{43}$$

of dimension $n \times n$. With the coordinate transformation

$$\boldsymbol{q} = \boldsymbol{R}\boldsymbol{p} \tag{44}$$

the equations of motion of a $\boldsymbol{M}$-$\boldsymbol{K}$-system (26) can be written in the form

$$\boldsymbol{M}\boldsymbol{R}\ddot{\boldsymbol{p}} + \boldsymbol{K}\boldsymbol{R}\boldsymbol{p} = \boldsymbol{0} \tag{45}$$

in the modal coordinates $\boldsymbol{p}$. Multiplication with $\boldsymbol{R}^\mathsf{T}$ from the left gives

$$\boldsymbol{R}^\mathsf{T}\boldsymbol{M}\boldsymbol{R}\ddot{\boldsymbol{p}} + \boldsymbol{R}^\mathsf{T}\boldsymbol{K}\boldsymbol{R}\boldsymbol{p} = \boldsymbol{0}. \tag{46}$$

Due to the orthogonality of the eigenvectors with respect to $\boldsymbol{M}$ and $\boldsymbol{K}$ the matrices

$$\tilde{\boldsymbol{M}} = \boldsymbol{R}^\mathsf{T}\boldsymbol{M}\boldsymbol{R} = \begin{bmatrix} \tilde{m}_1 & & & \boldsymbol{0} \\ & \tilde{m}_2 & & \\ & & \ddots & \\ \boldsymbol{0} & & & \tilde{m}_n \end{bmatrix}, \quad \tilde{\boldsymbol{K}} = \boldsymbol{R}^\mathsf{T}\boldsymbol{K}\boldsymbol{R} = \begin{bmatrix} \tilde{k}_1 & & & \boldsymbol{0} \\ & \tilde{k}_2 & & \\ & & \ddots & \\ \boldsymbol{0} & & & \tilde{k}_n \end{bmatrix}$$

are diagonal. The diagonal elements

$$\tilde{m}_i = \boldsymbol{r}_i^\mathsf{T}\boldsymbol{M}\boldsymbol{r}_i, \tag{47a}$$

$$\tilde{k}_i = \boldsymbol{r}_i^\mathsf{T}\boldsymbol{K}\boldsymbol{r}_i \tag{47b}$$

are called *modal mass* and *modal stiffness*. Both quantities of course depend on the normalization of the eigenvectors and moreover depend on an arbitrary common factor (since (46) can be multiplied by an arbitrary factors). The quotient

$$\omega_i = \sqrt{\frac{\tilde{k}_i}{\tilde{m}_i}} = \sqrt{\frac{\boldsymbol{r}_i^\mathsf{T}\boldsymbol{K}\boldsymbol{r}_i}{\boldsymbol{r}_i^\mathsf{T}\boldsymbol{M}\boldsymbol{r}_i}} = \operatorname{Im}\lambda_i, \qquad i = 1, \ldots, n \tag{48}$$