

**DYNAMICS OF
THE RIGID SOLID
WITH GENERAL
CONSTRAINTS
BY A MULTIBODY
APPROACH**

**NICOLAE PANDREA
NICOLAE-DORU STĂNESCU**

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DYNAMICS OF THE RIGID SOLID WITH GENERAL CONSTRAINTS BY A MULTIBODY APPROACH

Nicolae Pandrea and

Nicolae-Doru Stănescu

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Preface

This book deals with both holonomic and non-holonomic constraints to study the mechanics of the constrained rigid body. The approach is completely matrix and we study all types of the general constraints that may appear at a rigid solid. The discussion is performed in the most general case, not in particular cases defined by certain types of mechanisms. Our approach is a multibody type one and the obtaining of the matrix of constraints is highlighted in each case discussed in the book. In addition, algorithms for the numerical calculations are given for each type of constraint. The theory is applied to numerical examples which are completely solved, the diagrams resulted being also presented.

The book contains eight chapters as follows. The first chapter is an introduction presenting the elements of mathematical calculation that will be used in the book. The second chapter treats the kinematics of the rigid solid and in this chapter we obtain the distribution of velocities and accelerations for a rigid body. The next chapter is dedicated to the general theorem in the dynamics of the rigid solid, that is, the theorem of momentum, the theorem of the moment of momentum, and the kinetic energy; all these theorems are developed in matrix form. In the fourth chapter are presented the matrix differential equations of motion in the general case of the rigid solid with constraints; the equations of motion are obtained using the general theorems and using the Lagrange equations; a completely new proof is given for the equivalence of these two approaches. In the fifth chapter we discuss the equilibrium of the rigid solid; we introduce the generalized forces and their expressions; as a particular case we study the equilibrium of a rigid solid hanged by springs. The next chapter deals with the motion of the rigid solid having constraints at given proper points; we discuss the rigid body with one fixed point, the rigid body in rotational motion, the rigid body with one or several points situated on given surfaces or curves. In the seventh chapter we discuss the motion of the rigid solid with constraints on given proper curves; the chapter is

dedicated to the study of the rigid body at which given curves support on given curves or surfaces. The last chapter is dedicated to the motion of the rigid solid with constraints on the bounded surfaces; in this case the rigid body is supported at fixed points, or it rolls on curves or surfaces.

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This book is addressed to a large audience, to all those interested in using models and methods with holonomic and non-holonomic constraints in various fields like: mechanics, physics, civil and mechanical engineering, people involved in teaching, research or design, as well as students.

The book can be also used either as a stand-alone course for the master or PhD students, or as supplemental reading for the courses of computational mechanics, analytical mechanics, multibody mechanics etc. The prerequisites are the courses of elementary algebra and analysis, and mechanics.

Nicolae Pandrea and Nicolae-Doru Stănescu

Contents

Preface	xi
1 Elements of Mathematical Calculation	1
1.1 Vectors: Vector Operations	1
1.2 Real Rectangular Matrix	4
1.3 Square Matrix	6
1.4 Skew Matrix of Third Order	10
Further Reading	12
2 Kinematics of the Rigid Solid	15
2.1 Finite Displacements of the Points of Rigid Solid	15
2.2 Matrix of Rotation: Properties	16
2.2.1 <i>General Properties</i>	16
2.2.2 <i>Successive Displacements</i>	17
2.2.3 <i>Eigenvalues: Eigenvectors</i>	18
2.2.4 <i>The Expression of the Matrix of Rotation with the Aid of the Unitary Eigenvector and the Angle of Rotation</i>	20
2.2.5 <i>Symmetries: Decomposition of the Rotation into Two Symmetries</i>	24
2.2.6 <i>Rotations About the Axes of Coordinates</i>	25
2.3 Minimum Displacements: The Chasles Theorem	27
2.4 Small Displacements	33
2.5 Velocities of the Points of Rigid Solid	34

2.6	The Angular Velocity Matrix: Properties	37
2.6.1	<i>The Matrices of Rotation About the Axes of Coordinates</i>	37
2.6.2	<i>The Angular Velocity Matrix: The Angular Velocity Vector</i>	38
2.6.3	<i>The Matrix of the Partial Derivatives of the Angular Velocity</i>	39
2.7	Composition of the Angular Velocities	41
2.8	Accelerations of the Points of Rigid Solid	42
	Further Reading	43
3	General Theorems in the Dynamics of the Rigid Solid	45
3.1	Moments of Inertia	45
3.1.1	<i>Definitions: Relations Between the Moments of Inertia</i>	45
3.1.2	<i>Moments of Inertia for Homogeneous Rigid Solid Bodies</i>	47
3.1.3	<i>Centers of Weight</i>	47
3.1.4	<i>Variation of the Moments of Inertia Relative to Parallel Axes</i>	49
3.1.5	<i>Variation of the Moments of Inertia Relative to Concurrent Axes</i>	50
3.1.6	<i>Principal Axes of Inertia: Principal Moments of Inertia</i>	52
3.2	Momentum: The Theorem of Momentum	54
3.3	Moment of Momentum: The Theorem of Moment of Momentum	56
3.4	The Kinetic Energy of the Rigid Solid	57
	Further Reading	58
4	Matrix Differential Equations of the Motion of Rigid Solid	61
4.1	The Differential Equations Obtained from the General Theorems	61
4.1.1	<i>General Aspects</i>	61
4.1.2	<i>The Differential Equations</i>	62
4.2	The Lagrange Equations in the Case of the Holonomic Constraints	63
4.3	The Equivalence between the Differential Equations Obtained from the General Theorems and the Lagrange Equations	65
4.3.1	<i>The Equivalence for the First Component</i>	65
4.3.2	<i>The Equivalence for the Second Component</i>	66
4.4	The Matrix Differential Equations for the Motion of the Constrained Rigid Solid	71
4.4.1	<i>The Matrix of Constraints</i>	71
4.4.2	<i>The Lagrange Equations for Mechanical Systems with Constraints</i>	73
4.4.3	<i>The Mathematical Model of the Motion of Rigid Solid with Constraints</i>	75
4.4.4	<i>General Algorithm of Calculation</i>	76
4.4.5	<i>The Calculation of the Forces of Constraints</i>	78
4.4.6	<i>The Elimination of the Matrix of the Lagrange multipliers</i>	80
	Further Reading	85

5	Generalized Forces: The Equilibrium of the Rigid Solid	89
5.1	The Generalized Forces in the Case of a Mechanical System	89
5.2	The General Expressions of the Generalized Forces in the Case of Rigid Solid	90
5.2.1	<i>The Case When at a Point Acts a Given Force</i>	90
5.2.2	<i>The Case When the Rigid Solid is Acted by a Torque of Given Moment</i>	93
5.3	Conservative Forces	94
5.3.1	<i>General Aspects</i>	94
5.3.2	<i>The Weight</i>	96
5.3.3	<i>The Elastic Force of a Spring</i>	97
5.4	The Equilibrium of the Constrained Rigid Solid	98
5.4.1	<i>The Equations of Equilibrium: Numerical Solution</i>	98
5.4.2	<i>The Case When the Functions of Constraints Introduce Auxiliary Coordinates (Pseudo-Coordinates)</i>	100
5.5	The Equilibrium of the Heavy Rigid Solid Hanged by Springs	104
5.5.1	<i>The Matrix Equation of Equilibrium</i>	104
5.5.2	<i>Numerical Solution</i>	106
5.5.3	<i>The Case When the Fixed Reference System Coincides to the Local Reference System at the Equilibrium Position</i>	108
	Further Reading	109
6	The Motion of the Rigid Solid with Constraints at Given Proper Points	113
6.1	General Aspects: Classification	113
6.2	Mathematical Aspects: Notations	114
6.2.1	<i>The Case of the Motion Depending on Only the Generalized Coordinates $X_O, Y_O, Z_O, \psi, \theta, \varphi$</i>	114
6.2.2	<i>The Case of the Constraints Depending on the Pseudo-Coordinates Too</i>	115
6.2.3	<i>Relations of Calculation Necessary for the Numerical Algorithm</i>	115
6.3	The Study of the Rigid Solid with a Fixed Point	116
6.4	The Rigid Solid with Two Fixed Points (the Rotational Motion of the Rigid Solid)	118
6.5	The Rigid Solid with a Given Point Situated on a Fixed Surface	121
6.5.1	<i>The Case When the Surface is Defined by an Implicit Equation $F(X, Y, Z) = 0$</i>	121
6.5.2	<i>The Case When the Surface is Defined by Parametric Equations</i>	123
6.6	The Rigid Solid with Several Points Situated on Fixed Surfaces (Curves)	125
6.6.1	<i>The Case When the Surfaces are Defined by Implicit Equations</i>	125
6.6.2	<i>The Case When the Surfaces are Defined by Parametric Equations</i>	126

6.7	The Rigid Solid with a Fixed Point and with Another Point Situated on a Fixed Surface	127
6.7.1	<i>The Case When the Fixed Surface is Defined by an Implicit Equation</i>	127
6.7.2	<i>The Case When the Fixed Surface is Defined by Parametric Equations</i>	129
6.8	The Rigid Solid with Two Given Points Situated on a Fixed Curve	130
6.8.1	<i>The Case When the Curve is Defined by Two Implicit Equations</i>	130
6.8.2	<i>The Case When the Curve is Defined by Parametric Equations</i>	131
6.8.3	<i>The Helical Motion of the Rigid Solid</i>	132
	Further Reading	133
7	The Motion of the Rigid Solid with Constraints on Given Proper Curves	135
7.1	General Aspects: Classification	135
7.2	The Rigid Solid Supported at Fixed Points on Given Proper Curves	136
7.2.1	<i>Notations</i>	136
7.2.2	<i>The Matrix of Constraints</i>	137
7.3	The Rigid Solid at Which Given Proper Curves Support with Sliding on Fixed Curves	138
7.3.1	<i>Notations</i>	138
7.3.2	<i>The Simple Contact between the Curves</i>	139
7.3.3	<i>The Tangency Contact between Spatial Curves</i>	143
7.3.4	<i>Contact with Sliding between Planar Curves (Rolling with Sliding on the Plan)</i>	144
7.4	Rolling without Sliding of a Curve on a Fixed Curve	147
7.4.1	<i>The General Case for Spatial Curves</i>	147
7.4.2	<i>The Rolling Without Sliding of a Curve on a Fixed Curve in the Plan</i>	148
7.5	The Motion of the Rigid Solid at Which the Curves Jointed to It Support with Sliding on Fixed Surfaces	151
7.5.1	<i>The Case of a Single Curve</i>	151
7.5.2	<i>The Case of the Supporting with Sliding by Curves on Surface</i>	154
7.6	The Rolling without Sliding of a Disk Bounded by a Spatial Curve on a Fixed Surface	157
7.6.1	<i>The Matrix Differential Equation of Motion</i>	157
7.6.2	<i>The Forces at the Contact Point</i>	159
7.7	The Rolling without Sliding of a Planar Circle Disk on a Horizontal Plan	160
7.8	The Rolling without Sliding of a Planar Elliptic Disk on a Horizontal Plan	168

7.9	The Rolling without Sliding of a Hyperboidic Curve on a Horizontal Plan	175
7.9.1	<i>Hyperboidic Curves</i>	175
7.9.2	<i>The Matrix Differential Equation of Motion</i>	176
7.10	The Rolling without Sliding of a Planar Circle Disk on a Cylindrical Surface with Horizontal Generatrices	184
7.11	The Rolling without Sliding of Two Curves of a Rigid Solid on a Fixed Surface	192
7.11.1	<i>General Aspects</i>	192
7.11.2	<i>The Differential Equations of Motion</i>	195
7.11.3	<i>The Algorithm of Numerical Calculation</i>	196
7.12	The Rolling without Sliding of an Axle with Wheels (Disks) with Angular Deviations on a Horizontal Plan	197
7.13	The Rolling without Sliding of an Axle with Disks on a Hyperbolic Paraboloid	204
7.13.1	<i>General Aspects</i>	204
7.13.2	<i>The Initial Position</i>	206
7.13.3	<i>The Differential Equations</i>	207
	Further Reading	214
8	The Motion of the Rigid Solid with Constraints on the Bounding Surface	217
8.1	General Aspects: Classification	217
8.2	The Rigid Solid Supported at Fixed Points	218
8.2.1	<i>The Matrix of Constraints</i>	218
8.2.2	<i>The Matrix Differential Equation of Motion</i>	220
8.2.3	<i>The Algorithm of Calculation</i>	221
8.3	The Rigid Solid Supported with Sliding on Fixed Curves	236
8.3.1	<i>The Matrix of Constraints</i>	236
8.3.2	<i>The Matrix Differential Equation of Motion</i>	239
8.3.3	<i>The Reactions</i>	239
8.3.4	<i>The Algorithm of Calculation</i>	240
8.4	The Rolling without Sliding of the Rigid Solid on Two Fixed Curves	244
8.4.1	<i>General Considerations</i>	244
8.4.2	<i>The Differential Equations of Motion</i>	246
8.4.3	<i>The Algorithm for the Numerical Calculation</i>	248
8.5	The Rolling without Sliding of a Rigid Solid on a Fixed Surface	254
8.5.1	<i>The Matrix of Constraints</i>	254
8.5.2	<i>The Matrix Differential Equation of Motion</i>	256
8.6	The Rolling without Sliding of a Toroidal Wheel on a Horizontal Plan	257
8.6.1	<i>The Equations of Torus</i>	257

8.6.2	<i>The Tangency Conditions</i>	258
8.6.3	<i>The Initial Conditions</i>	258
8.6.4	<i>The Differential Equations of Motion</i>	260
8.7	The Rolling without Sliding of a Rigid Solid Supported on Two Fixed Surfaces	265
8.7.1	<i>General Aspects</i>	265
8.7.2	<i>The Differential Equations of Motion</i>	267
8.7.3	<i>The Determination of the Forces of Constraints</i>	269
8.7.4	<i>The Rolling without Sliding of an Ellipsoid Acted only by its Own Weight on Two Plans</i>	270
8.8	The Rolling without Sliding of a Rigid Solid Supported at Two Points on a Fixed Surface	291
8.8.1	<i>General Aspects</i>	291
8.8.2	<i>The Differential Equations of Motion: The Calculation of the Forces of Constraints</i>	293
	Further Reading	294
	Appendix	297
	Index	315

1

Elements of Mathematical Calculation

This chapter is an introduction presenting the elements of mathematical calculation that will be used in the book.

1.1 Vectors: Vector Operations

A *vector* (denoted by \mathbf{a}) is defined by its numerical magnitude or modulus $|\mathbf{a}|$, by the direction Δ , and by sense. The vector is represented (Fig. 1.1) by an orientated segment of straight line.

The sum of two vectors \mathbf{a} , \mathbf{b} is the vector \mathbf{c} (Fig. 1.2) represented by the diagonal of the parallelogram constructed on the two vectors; it reads

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \quad (1.1)$$

The *unit vector* \mathbf{u} of the vector \mathbf{a} (or of the direction Δ) is defined by the relation

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}. \quad (1.2)$$

If one denotes by \mathbf{i} , \mathbf{j} , \mathbf{k} the unit vectors of the axes of dextrorsum orthogonal reference system $Oxyz$, and by a_x , a_y , a_z the projections of vector \mathbf{a} onto the axes, then one may write the analytical expression

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}. \quad (1.3)$$

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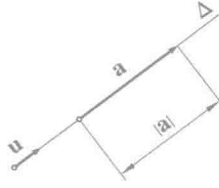


Figure 1.1 Representation of a vector.

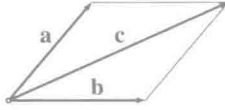


Figure 1.2 The sum of two vectors.

The scalar (dot) product of two vectors is defined by the expression

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \alpha, \quad (1.4)$$

where α is the angle between the two vectors.

We obtain the equalities

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1 \quad (1.5)$$

and, consequently, one deduces the analytical expressions

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z, \quad (1.6)$$

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \quad |\mathbf{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}, \quad (1.7)$$

$$\cos \alpha = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}}. \quad (1.8)$$

The vector (cross) product of two vectors, denoted by \mathbf{c} ,

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}, \quad (1.9)$$

is the vector perpendicular onto the plan of the vectors \mathbf{a} and \mathbf{b} , while the sense is given by the rule of the right screw when the vector \mathbf{a} rotates over the vector \mathbf{b} (making the smallest angle); the modulus has the expression

$$|\mathbf{c}| = |\mathbf{a}| |\mathbf{b}| \sin \alpha, \quad (1.10)$$

α being the smallest angle between the vectors \mathbf{a} and \mathbf{b} .

One obtains the equalities

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad (1.11)$$

and the analytical expression

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}. \quad (1.12)$$

The mixed product of three vectors, defined by the relation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and denoted by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, leads to the successive equalities

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \quad (1.13)$$

The mixed product $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is equal to the volume with sign of the parallelepiped constructed having the three vectors as edges (Fig. 1.3). It is equal to zero if and only if the three vectors are coplanar.

The double vector product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ satisfies the equality

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}. \quad (1.14)$$

The reciprocal vectors of the (non-coplanar) vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are defined by the expressions

$$\mathbf{a}^* = \frac{\mathbf{b} \times \mathbf{c}}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \mathbf{b}^* = \frac{\mathbf{c} \times \mathbf{a}}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \mathbf{c}^* = \frac{\mathbf{a} \times \mathbf{b}}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \quad (1.15)$$

and satisfy the equality

$$(\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*) = \frac{1}{(\mathbf{a}, \mathbf{b}, \mathbf{c})}. \quad (1.16)$$

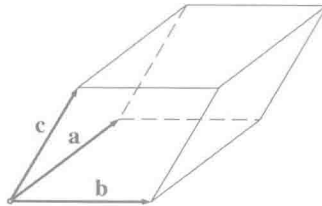


Figure 1.3 The geometric interpretation of the mixed product of three vectors.

An arbitrary vector \mathbf{v} may be written in the form

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a}^*)\mathbf{a} + (\mathbf{v} \cdot \mathbf{b}^*)\mathbf{b} + (\mathbf{v} \cdot \mathbf{c}^*)\mathbf{c}, \quad (1.17)$$

or as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a})\mathbf{a}^* + (\mathbf{v} \cdot \mathbf{b})\mathbf{b}^* + (\mathbf{v} \cdot \mathbf{c})\mathbf{c}^*. \quad (1.18)$$

1.2 Real Rectangular Matrix

By *real rectangular matrix* we understand a table with m rows and n columns ($m \neq n$)

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad (1.19)$$

where the *elements* a_{ij} are real numbers.

Sometimes, we use the abridged notation

$$[\mathbf{A}] = (a_{ij}) \text{ or } [\mathbf{A}] = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}. \quad (1.20)$$

The *multiplication between a matrix and a scalar* $\lambda \in \mathbb{R}$ is defined by the relation

$$\lambda[\mathbf{A}] = (\lambda a_{ij}), \quad (1.21)$$

while the *sum of two matrices of the same type* (with the same number of rows and the same number of columns) is defined by

$$[\mathbf{A}] + [\mathbf{B}] = (a_{ij} + b_{ij}). \quad (1.22)$$

The *zero matrix* or the *null matrix* is the matrix denoted by $[\mathbf{0}]$, which has all its elements equal to zero.

The zero matrix verifies the relations

$$[\mathbf{A}] + [\mathbf{0}] = [\mathbf{0}] + [\mathbf{A}] = [\mathbf{A}]. \quad (1.23)$$

The *transpose matrix* $[\mathbf{A}]^T$ is the matrix obtained transforming the rows of the matrix $[\mathbf{A}]$ into columns, that is

$$[\mathbf{A}]^T = (a_{ji}). \quad (1.24)$$

The transposing operation has the following properties

$$[[\mathbf{A}]^T]^T = [\mathbf{A}], [[\mathbf{A}] + [\mathbf{B}]]^T = [\mathbf{A}]^T + [\mathbf{B}]^T, \quad (1.25)$$

where we assumed that the sum can be performed.

The matrix with one column bears the name *column matrix* or *column vector* and it is denoted by $\{\mathbf{A}\}$, that is

$$\{\mathbf{A}\} = [a_{11} \ a_{21} \ \dots \ a_{m1}]^T, \quad (1.26)$$

while the matrix with one row is called *row matrix* or *row vector* and is denoted as

$$[\mathbf{A}] = [a_{11} \ a_{12} \ \dots \ a_{1n}], \quad (1.27)$$

or

$$[\mathbf{A}] = \{\mathbf{A}\}^T, \quad (1.28)$$

where

$$\{\mathbf{A}\} = [a_{11} \ a_{12} \ \dots \ a_{1n}]^T. \quad (1.29)$$

If the matrix $[\mathbf{A}]$ has m rows and n columns, and the matrix $[\mathbf{B}]$ has n rows and p columns, *then* the two matrices *can be multiplied* and the result is a matrix $[\mathbf{C}]$ with m rows and p columns

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}], \quad (1.30)$$

where the elements c_{ij} , $1 \leq i \leq m$, $1 \leq j \leq p$, of the matrix $[\mathbf{C}]$ satisfy the equality

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}, \quad (1.31)$$

that is, the elements of the product matrix are obtained by multiplying the rows of matrix $[\mathbf{A}]$ by the columns of matrix $[\mathbf{B}]$.

The transpose of the product matrix is given by the relation

$$[[\mathbf{A}][\mathbf{B}]]^T = [\mathbf{B}]^T [\mathbf{A}]^T. \quad (1.32)$$

In some cases, there may exist *matrices of matrices* and the multiplication is performed as in the following example

$$\begin{bmatrix} [\mathbf{A}_1] & [\mathbf{A}_2] \\ [\mathbf{A}_3] & [\mathbf{A}_4] \\ [\mathbf{A}_5] & [\mathbf{A}_6] \end{bmatrix} \begin{bmatrix} [\mathbf{B}_1] & [\mathbf{B}_2] \\ [\mathbf{B}_3] & [\mathbf{B}_4] \end{bmatrix} = \begin{bmatrix} [\mathbf{A}_1][\mathbf{B}_1] + [\mathbf{A}_2][\mathbf{B}_3] & [\mathbf{A}_1][\mathbf{B}_2] + [\mathbf{A}_2][\mathbf{B}_4] \\ [\mathbf{A}_3][\mathbf{B}_1] + [\mathbf{A}_4][\mathbf{B}_3] & [\mathbf{A}_3][\mathbf{B}_2] + [\mathbf{A}_4][\mathbf{B}_4] \\ [\mathbf{A}_5][\mathbf{B}_1] + [\mathbf{A}_6][\mathbf{B}_3] & [\mathbf{A}_5][\mathbf{B}_2] + [\mathbf{A}_6][\mathbf{B}_4] \end{bmatrix}, \quad (1.33)$$

where we assumed that the operations of multiplication and addition of matrices can be performed for each separate case.

1.3 Square Matrix

The matrix $[\mathbf{A}]$ is a *square matrix* if the number of rows is equal to the number of columns; hence

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad (1.34)$$

where the number n is the *dimension* or the *order* of the matrix.

The *determinant* associated to the matrix $[\mathbf{A}]$ is denoted by $\det[\mathbf{A}]$.

If $[\mathbf{A}_{ij}]$ is the matrix obtained from the matrix $[\mathbf{A}]$ by the suppression of the row i and the column j , then the *algebraic complement* a_{ij}^* is given by the expression

$$a_{ij}^* = (-1)^{i+j} \det[\mathbf{A}_{ij}], \quad 1 \leq i, j \leq n, \quad (1.35)$$

and the following relation holds true

$$\sum_{k=1}^n a_{ik} a_{jk}^* = \sum_{k=1}^n a_{kj} a_{ki}^* = \begin{cases} 0 & \text{for } i \neq j \\ \det[\mathbf{A}] & \text{for } i = j \end{cases}. \quad (1.36)$$

The determinants of the matrices satisfy the equalities

$$\det[\mathbf{A}] = \det[\mathbf{A}]^T, \quad (1.37)$$

$$\det[[\mathbf{A}][\mathbf{B}]] = \det[\mathbf{A}] \cdot \det[\mathbf{B}], \quad (1.38)$$