DYNAMICS OF THE RIGID SOLID WITH GENERAL CONSTRAINTS BY A MULTIBODY APPROACH

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Preface

This book deals with both holonomic and non-holonomic constraints to study the mechanics of the constrained rigid body. The approach is completely matrix and we study all types of the general constraints that may appear at a rigid solid. The discussion is performed in the most general case, not in particular cases defined by certain types of mechanisms. Our approach is a multibody type one and the obtaining of the matrix of constraints is highlighted in each case discussed in the book. In addition, algorithms for the numerical calculations are given for each type of constraint. The theory is applied to numerical examples which are completely solved, the diagrams resulted being also presented.

The book contains eight chapters as follows. The first chapter is an introduction presenting the elements of mathematical calculation that will be used in the book. The second chapter treats the kinematics of the rigid solid and in this chapter we obtain the distribution of velocities and accelerations for a rigid body. The next chapter is dedicated to the general theorem in the dynamics of the rigid solid, that is, the theorem of momentum, the theorem of the moment of momentum, and the kinetic energy; all these theorems are developed in matrix form. In the fourth chapter are presented the matrix differential equations of motion in the general case of the rigid solid with constraints; the equations of motion are obtained using the general theorems and using the Lagrange equations; a completely new proof is given for the equivalence of these two approaches. In the fifth chapter we discuss the equilibrium of the rigid solid; we introduce the generalized forces and their expressions; as a particular case we study the equilibrium of a rigid solid hanged by springs. The next chapter deals with the motion of the rigid solid having constraints at given proper points; we discuss the rigid body with one fixed point, the rigid body in rotational motion, the rigid body with one or several points situated on given surfaces or curves. In the seventh chapter we discuss the motion of the rigid solid with constraints on given proper curves; the chapter is dedicated to the study of the rigid body at which given curves support on given curves or surfaces. The last chapter is dedicated to the motion of the rigid solid with constraints on the bounded surfaces; in this case the rigid body is supported at fixed points, or it rolls on curves or surfaces.

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This book is addressed to a large audience, to all those interested in using models and methods with holonomic and non-holonomic constraints in various fields like: mechanics, physics, civil and mechanical engineering, people involved in teaching, research or design, as well as students.

The book can be also used either as a stand-alone course for the master or PhD students, or as supplemental reading for the courses of computational mechanics, analytical mechanics, multibody mechanics etc. The prerequisites are the courses of elementary algebra and analysis, and mechanics.

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Elements of Mathematical Calculation

This chapter is an introduction presenting the elements of mathematical calculation that will be used in the book.

1.1 Vectors: Vector Operations

A *vector* (denoted by $\mathbf{a}$) is defined by its numerical magnitude or modulus $|\mathbf{a}|$, by the direction Δ, and by sense. The vector is represented (Fig. 1.1) by an orientated segment of straight line.

The sum of two vectors $\mathbf{a}$, $\mathbf{b}$ is the vector $\mathbf{c}$ (Fig. 1.2) represented by the diagonal of the parallelogram constructed on the two vectors; it reads

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{1.1}$$

The unit vector $\mathbf{u}$ of the vector $\mathbf{a}$ (or of the direction Δ) is defined by the relation

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{1.2}$$

If one denotes by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors of the axes of dextrorsum orthogonal reference system $Oxyz$, and by a_x, a_y, a_z the projections of vector $\mathbf{a}$ onto the axes, then one may write the analytical expression

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \tag{1.3}$$

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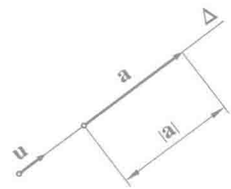

Figure 1.1 Representation of a vector.

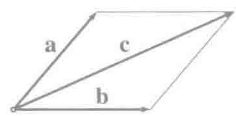

Figure 1.2 The sum of two vectors.

The scalar (dot) product of two vectors is defined by the expression

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\alpha, \tag{1.4}$$

where α is the angle between the two vectors.

We obtain the equalities

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0,\ \mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1 \tag{1.5}$$

and, consequently, one deduces the analytical expressions

$$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z, \tag{1.6}$$

$$|\mathbf{a}|=\sqrt{a_x^2+a_y^2+a_z^2},\ |\mathbf{b}|=\sqrt{b_x^2+b_y^2+b_z^2}, \tag{1.7}$$

$$\cos\alpha=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}. \tag{1.8}$$

The vector (cross) product of two vectors, denoted by **c**,

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}, \tag{1.9}$$

is the vector perpendicular onto the plan of the vectors **a** and **b**, while the sense is given by the rule of the right screw when the vector **a** rotates over the vector **b** (making the smallest angle); the modulus has the expression

$$|\mathbf{c}|=|\mathbf{a}||\mathbf{b}|\sin\alpha, \tag{1.10}$$

α being the smallest angle between the vectors **a** and **b**.
One obtains the equalities

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}, \tag{1.11}$$

and the analytical expression

$$\mathbf{a}\times\mathbf{b}=\left(a_yb_z-a_zb_y\right)\mathbf{i}+\left(a_zb_x-a_xb_z\right)\mathbf{j}+\left(a_xb_y-a_yb_x\right)\mathbf{k}. \tag{1.12}$$

The mixed product of three vectors, defined by the relation $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and denoted by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, leads to the successive equalities

$$(\mathbf{a},\mathbf{b},\mathbf{c})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.13}$$

The mixed product $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is equal to the volume with sign of the parallelepiped constructed having the three vectors as edges (Fig. 1.3). It is equal to zero if and only if the three vectors are coplanar.

The double vector product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ satisfies the equality

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{1.14}$$

The reciprocal vectors of the (non-coplanar) vectors **a**, **b**, **c** are defined by the expressions

$$\mathbf{a}^*=\frac{\mathbf{b}\times\mathbf{c}}{(\mathbf{a},\mathbf{b},\mathbf{c})},\ \mathbf{b}^*=\frac{\mathbf{c}\times\mathbf{a}}{(\mathbf{a},\mathbf{b},\mathbf{c})},\ \mathbf{c}^*=\frac{\mathbf{a}\times\mathbf{b}}{(\mathbf{a},\mathbf{b},\mathbf{c})}, \tag{1.15}$$

and satisfy the equality

$$(\mathbf{a}^*,\mathbf{b}^*,\mathbf{c}^*)=\frac{1}{(\mathbf{a},\mathbf{b},\mathbf{c})}. \tag{1.16}$$

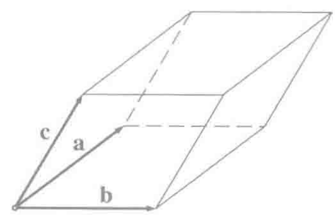

Figure 1.3 The geometric interpretation of the mixed product of three vectors.

An arbitrary vector **v** may be written in the form

$$\mathbf{v}=(\mathbf{v}\cdot\mathbf{a}^*)\mathbf{a}+(\mathbf{v}\cdot\mathbf{b}^*)\mathbf{b}+(\mathbf{v}\cdot\mathbf{c}^*)\mathbf{c}, \tag{1.17}$$

or as

$$\mathbf{v}=(\mathbf{v}\cdot\mathbf{a})\mathbf{a}^*+(\mathbf{v}\cdot\mathbf{b})\mathbf{b}^*+(\mathbf{v}\cdot\mathbf{c})\mathbf{c}^*. \tag{1.18}$$

1.2 Real Rectangular Matrix

By *real rectangular matrix* we understand a table with m rows and n columns ($m \neq n$)

$$[\mathbf{A}]=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \tag{1.19}$$

where the *elements* a_{ij} are real numbers.

Sometimes, we use the abridged notation

$$[\mathbf{A}]=(a_{ij}) \text{ or } [\mathbf{A}]=(a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}. \tag{1.20}$$

The multiplication between a matrix and a scalar $\lambda \in \mathbb{R}$ is defined by the relation

$$\lambda[\mathbf{A}]=(\lambda a_{ij}), \tag{1.21}$$

while *the sum of two matrices of the same type* (with the same number of rows and the same number of columns) is defined by

$$[\mathbf{A}]+[\mathbf{B}]=(a_{ij}+b_{ij}). \tag{1.22}$$

The zero matrix or the null matrix is the matrix denoted by $[\mathbf{0}]$, which has all its elements equal to zero.

The zero matrix verifies the relations

$$[\mathbf{A}]+[\mathbf{0}]=[\mathbf{0}]+[\mathbf{A}]=[\mathbf{A}]. \tag{1.23}$$

The transpose matrix $[\mathbf{A}]^{\mathrm{T}}$ is the matrix obtained transforming the rows of the matrix $[\mathbf{A}]$ into columns, that is

$$[\mathbf{A}]^{\mathrm{T}} = (a_{ji}). \tag{1.24}$$

The transposing operation has the following properties

$$\left[[\mathbf{A}]^{\mathrm{T}}\right]^{\mathrm{T}} = [\mathbf{A}], [[\mathbf{A}] + [\mathbf{B}]]^{\mathrm{T}} = [\mathbf{A}]^{\mathrm{T}} + [\mathbf{B}]^{\mathrm{T}}, \tag{1.25}$$

where we assumed that the sum can be performed.

The matrix with one column bears the name *column matrix* or *column vector* and it is denoted by $\{\mathbf{A}\}$, that is

$$\{\mathbf{A}\} = [a_{11} \;\; a_{21} \;\; \ldots \;\; a_{m1}]^{\mathrm{T}}, \tag{1.26}$$

while the matrix with one row is called *row matrix* or *row vector* and is denoted as

$$[\mathbf{A}] = [a_{11} \;\; a_{12} \;\; \ldots \;\; a_{1n}], \tag{1.27}$$

or

$$[\mathbf{A}] = \{\mathbf{A}\}^{\mathrm{T}}, \tag{1.28}$$

where

$$\{\mathbf{A}\} = [a_{11} \;\; a_{12} \;\; \ldots \;\; a_{1n}]^{\mathrm{T}}. \tag{1.29}$$

If the matrix $[\mathbf{A}]$ has m rows and n columns, and the matrix $[\mathbf{B}]$ has n rows and p columns, *then* the two matrices *can be multiplied* and the result is a matrix $[\mathbf{C}]$ with m rows and p columns

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}], \tag{1.30}$$

where the elements c_{ij}, $1 \le i \le m$, $1 \le j \le p$, of the matrix $[\mathbf{C}]$ satisfy the equality

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \tag{1.31}$$

that is, the elements of the product matrix are obtained by multiplying the rows of matrix $[\mathbf{A}]$ by the columns of matrix $[\mathbf{B}]$.

The transpose of the product matrix is given by the relation

$$[[\mathbf{A}][\mathbf{B}]]^{\mathrm{T}} = [\mathbf{B}]^{\mathrm{T}}[\mathbf{A}]^{\mathrm{T}}. \tag{1.32}$$

In some cases, there may exist *matrices of matrices* and the multiplication is performed as in the following example

$$\begin{bmatrix} [\mathbf{A}_1] & [\mathbf{A}_2] \\ [\mathbf{A}_3] & [\mathbf{A}_4] \\ [\mathbf{A}_5] & [\mathbf{A}_6] \end{bmatrix} \begin{bmatrix} [\mathbf{B}_1] & [\mathbf{B}_2] \\ [\mathbf{B}_3] & [\mathbf{B}_4] \end{bmatrix} = \begin{bmatrix} [\mathbf{A}_1][\mathbf{B}_1] + [\mathbf{A}_2][\mathbf{B}_3] & [\mathbf{A}_1][\mathbf{B}_2] + [\mathbf{A}_2][\mathbf{B}_4] \\ [\mathbf{A}_3][\mathbf{B}_1] + [\mathbf{A}_4][\mathbf{B}_3] & [\mathbf{A}_3][\mathbf{B}_2] + [\mathbf{A}_4][\mathbf{B}_4] \\ [\mathbf{A}_5][\mathbf{B}_1] + [\mathbf{A}_6][\mathbf{B}_3] & [\mathbf{A}_5][\mathbf{B}_2] + [\mathbf{A}_6][\mathbf{B}_4] \end{bmatrix}, \tag{1.33}$$

where we assumed that the operations of multiplication and addition of matrices can be performed for each separate case.

1.3 Square Matrix

The matrix $[\mathbf{A}]$ is a *square matrix* if the number of rows is equal to the number of columns; hence

$$[\mathbf{A}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \tag{1.34}$$

where the number n is *the dimension* or *the order of the matrix*.

The determinant associated to the matrix $[\mathbf{A}]$ is denoted by $\det[\mathbf{A}]$.

If $[\mathbf{A}_{ij}]$ is the matrix obtained from the matrix $[\mathbf{A}]$ by the suppression of the row i and the column j, then *the algebraic complement* a_{ij}^* is given by the expression

$$a_{ij}^* = (-1)^{i+j} \det\left[\mathbf{A}_{ij}\right], 1 \le i, j \le n, \tag{1.35}$$

and the following relation holds true

$$\sum_{k=1}^{n} a_{ik} a_{jk}^* = \sum_{k=1}^{n} a_{kj} a_{ki}^* = \begin{cases} 0 & \text{for } i \neq j \\ \det[\mathbf{A}] & \text{for } i = j \end{cases}. \tag{1.36}$$

The determinants of the matrices satisfy the equalities

$$\det[\mathbf{A}] = \det[\mathbf{A}]^{\mathrm{T}}, \tag{1.37}$$

$$\det[[\mathbf{A}][\mathbf{B}]] = \det[\mathbf{A}] \cdot \det[\mathbf{B}], \tag{1.38}$$