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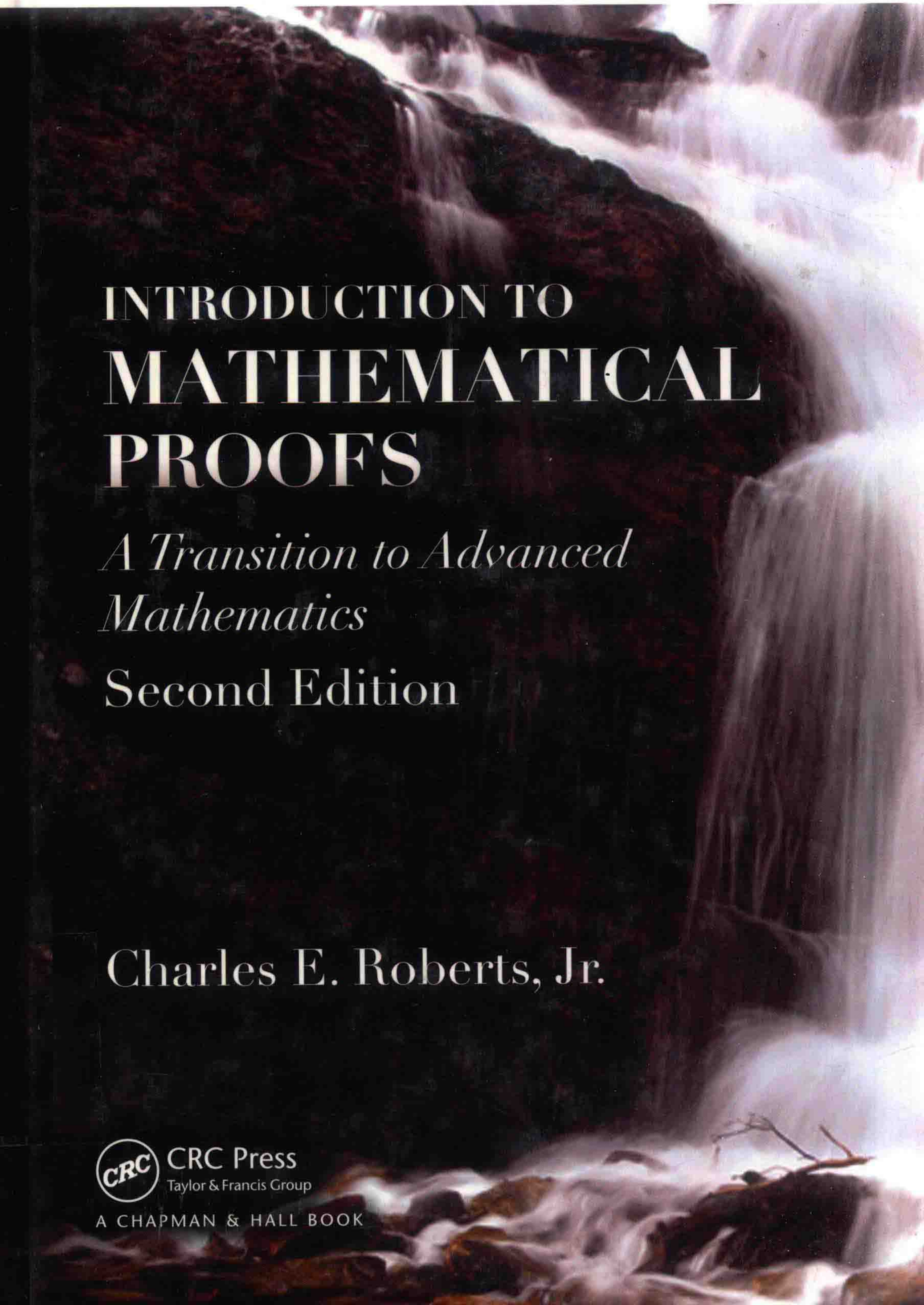

INTRODUCTION TO MATHEMATICAL PROOFS

A Transition to Advanced Mathematics

Second Edition

Charles E. Roberts, Jr.

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

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Series Editors: Al Boggess and Ken Rosen

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To

My wife, Imogene

My children, Eric and Natalie

and

My grandsons, Tristan and Luke

Preface to the Second Edition

First, I wish to thank my students, colleagues, and readers for indicating where they thought additional examples would be beneficial and for suggesting more topics to include in the text. Those suggestions have led to the inclusion of several new examples and topics in the second edition. Additional examples were included in the text to help students better understand concepts which are being introduced to them for the first time. For instance, an example was included in the text of a proof of the triangle inequality by cases which requires four cases.

In order to help students know exactly where proofs to lemmas and proofs to theorems end and where examples end, I have placed at the end of proofs and examples the black square symbol ■.

Certainly, the most noticeable change between the second edition and the first edition, is the inclusion of more than 125 new exercises in sections titled More Challenging Exercises. These sections appear in the review exercises at the end of each of the first seven chapters. Students who like to be challenged should benefit greatly from these exercises. For instance, in the More Challenging Exercises at the end of Chapter 1 new topics such as the binary operators nand, nor, and the Sheffer stroke are introduced and explored. In Chapter 2 in the section More Challenging Exercises the Fano plane is examined and problems of the type "Statement. Theorem? Proof?" are introduced for the first time. Problems of this kind also appear in Chapters 3 through 6. In these problems, a statement is given, and the student is to decide if the statement is true or false. If the statement is true, it is a theorem, and the student must then determine if the given proof is valid or invalid. If the proof is valid, the student is done. If the proof is invalid, the student is asked to produce a valid proof. If the statement is false, then, of course, the proof is invalid.

I thank my editor, Robert Ross, for his able assistance, guidance, and encouragement throughout the revision process. I thank Ken Rosen for his very insightful comments and suggestions. And finally, I thank the production staff at Taylor & Francis/CRC Press for their efforts in developing this second edition.

Charles Roberts

Preface

This text is written for undergraduate mathematics majors and minors who have previously taken only computationally oriented, problem solving mathematics courses. Usually these students are freshmen and sophomores. The primary objectives of the text are to teach the reader (1) to reason logically, (2) to read the proofs of others critically, and (3) to write valid mathematical proofs. We intend to help students develop the skills necessary to write correct, clear, and concise proofs. Ultimately, we endeavor to prepare students to succeed in more advanced mathematics courses such as abstract algebra, analysis, and geometry where they are expected to write proofs and construct counterexamples instead of performing computations and solving problems. The aim of the text is to facilitate a smooth transition from courses designed to develop computational skills and problem solving abilities to courses which emphasize theorem proving.

Logic is presented in Chapter 1, because logic is the underlying language of mathematics, because logic is the basis of all reasoned argument, and because logic developed earliest historically. This text may well be the only place in the undergraduate mathematics curriculum where a student is introduced to the study of logic. Knowing logic should benefit students not only in future mathematics courses but in other facets of their lives as well. Formal proofs are included, because each step in a formal proof requires a justification. And students need to understand that when they write an informal proof, each statement should be justified unless the justification is apparent to the reader.

In Chapter 2, deductive mathematical systems are defined and discussed. Various proof techniques are presented, and each proof technique is illustrated with several examples. Some theorems are proved using more than one proof technique, so that the reader may compare and contrast the techniques. The role of conjectures in mathematics is introduced, and proof and disproof of conjectures are explored. Interesting conjectures which recently have been proved true or disproved, and conjectures which still remain open are stated and discussed. The integers and their properties are developed from the axioms and properties of the natural numbers; the rational numbers and their properties are derived from the integers; and, finally, the method for developing the system of real numbers from the rational numbers is described.

Elementary topics in set theory are presented in Chapter 3. A thorough understanding of basic set theory is necessary for success in advanced mathematics courses. In addition, using set notation promotes precision and clarity when communicating mathematical ideas.

Relations and functions play a major role in many branches of mathematics and the sciences. Therefore, in Chapters 4 and 5, relations and functions are defined and their various properties are examined in detail.

In Chapter 6, proof by mathematical induction, in its various forms, is introduced and several theorems are proved using induction.

The last three chapters, which are optional, introduce the reader to the concept of cardinalities of sets (Chapter 7), and to the concepts and proofs in real analysis (Chapter 8) and in group theory (Chapter 9).

The appendix discusses reading and writing proofs and includes some basic guidelines to follow when writing proofs. We encourage students to read the appendix more than once during the semester and to use it as a reference when writing proofs.

Several different syllabi can be designed for this text depending upon the previous preparation and mathematical maturity of the students and the goals, objectives, and preferences of the instructor. Chapters 1 through 6 constitute the core of the course we teach during one semester. When time permits, we present some additional topics from Chapters 7, 8, and 9.

Features of the Text. This text is written in a friendly, conversational style, yet it maintains the proper level of mathematical rigor. Most sections are of appropriate length for presentation in one lecture session. Several biographical sketches and historical comments have been included to enrich and enliven the text. Generally, mathematics is presented as a continually evolving discipline, and the material presented should fulfill the needs of students with a wide range of backgrounds. Numerous technical terms which the student will encounter in more advanced courses are defined and illustrated. Many theorems from different disciplines in mathematics and of varying degrees of complexity are stated and proved. Numerous examples illustrate in detail how to write proofs and show how to solve problems. These examples serve as models for students to emulate when solving exercises. Exercises of varying difficulty appear at the end of each section.

Acknowledgments. This text evolved from lecture notes for a course which I have taught at Indiana State University for a number of years. I would like to thank my students and my colleagues for their support, encouragement, and constructive criticisms. Also, I would like to thank my editor Robert Stern and my project coordinator Stephanie Morkert of Taylor & Francis/CRC Press for their assistance in bringing this text to fruition.

Charles Roberts
2010

Contents

Chapter 1

Logic

There are many definitions of logic; however, we will consider logic to be the study of the methods and principles used to distinguish valid reasoning from invalid reasoning. Logic is a part of mathematics; moreover, in a broad sense, it is the language of mathematics.

In this chapter, we will study elementary symbolic logic. Logic is the basis of all reasoned argument, and therefore logic is the basis for valid mathematical proofs. The study of logic as a body of knowledge in Western Civilization originated with Aristotle (384–322 B.C.), one of the greatest philosophers of ancient Greece. He was a student of Plato for twenty years (from 367 to 347 B.C., when Plato died). Later, Aristotle tutored Alexander the Great, and in 334 B.C. he founded his own school of philosophy in the Lyceum. After his death, Aristotle's writings on reasoning were collected together in a body of work called the *Organon*. The contents of the *Organon* is the basis for the subject of logic, although the word "logic" did not acquire its current meaning until the second century A.D. The word "logic" is a derivative of the Greek word *logos,* which translates into English as "word," "speech," or "reason."

Aristotle was the first to develop rules for correct reasoning. However, he expressed logic in ordinary language, and, consequently, it was subject to the ambiguities of natural language. At an early age, the German philosopher, mathematician, and logician Gottfried Wilhelm Leibniz (1646–1716) was not satisfied with Aristotelian logic and began to develop his own ideas. He had a lifelong goal of developing a universal language and a calculus of reasoning. His idea was that the principles of reasoning could be reduced to a formal symbolic system in which controversies (not just mathematical ones) could be settled by calculations. Thus, Leibniz envisioned an algebra or calculus of thought. He made some strides toward his goal, but his work was largely forgotten.

The English mathematician and logician August De Morgan (1806–1871) presented ideas for improving classical logic in the 1840s. The key ideas he contributed in his text *Formal Logic* (1847) include the introduction of the concept of a universe of discourse; names for contraries; disjunction, conjunction, and negation of propositions; abbreviated notation for propositions; compound names; and notation for syllogisms. De Morgan intended to improve the syllogism and use it as the main device in reasoning. In order to ensure there were names for the contraries of compound names, he stated the famous De Morgan Laws. By creating some of the most basic concepts of modern logic, De Morgan contributed substantially to the change that was taking place in logic in the mid-1800s. However, his notational system was viewed as too complex, so he received little credit for the development of modern logic.

The English mathematician George Boole (1815–1864) is generally credited with founding the modern algebra of logic and hence symbolic logic. At the age of sixteen, Boole was an assistant teacher. In 1835, he opened his own school and began to study mathematics on his own. He never attended an institution of higher learning. He taught himself all of the higher mathematics he knew. In 1840, he began to publish papers on analysis in the *Cambridge Mathematical Journal*. In 1847, Boole published the text *The Mathematical Analysis of Logic*. Initially, Boole wanted to express all the statements of classical logic as equations and then apply algebraic transformations to derive the known valid arguments of logic. Near the end of writing the text, Boole realized that his algebra of logic applied to any finite collection of premises with any number of symbols. Boole's logic was limited to what is presently called the **propositional calculus**. It is the propositional calculus we will study in this chapter.

1.1 Statements, Negation, and Compound Statements

In the English language, sentences are classified according to their usage. A **declarative sentence** makes a statement. An **imperative sentence** gives a command or makes a request. An **interrogative sentence** asks a question. And an **exclamatory sentence** expresses strong feeling. Consider the following sentences:

1. Indianapolis is the capital of Indiana.
2. Tell Tom I will be home later.
3. What time is it?
4. I wish you were here!

The first sentence is declarative, the second sentence is imperative, the third sentence is interrogative, and the fourth sentence is exclamatory. However, the same sentence can be written to be declarative, interrogative, or exclamatory. For instance,

We won the game. [declarative]

We won the game? [interrogative]

We won the game! [exclamatory]

The declarative sentence "Indianapolis is the capital of Indiana" is "true" while the sentence "Minneapolis is the capital of Indiana" is "false."

Symbolic logic applies only to special declarative sentences which are called statements or propositions.

A **statement** or **proposition** is a declarative sentence that is either true or false, but not both true and false.

The terms true and false are left undefined, but it is assumed that their meaning is intuitively understood. Some declarative sentences might be true or false depending on the context or circumstance. Such sentences are not considered to be statements. For example, the sentences "She is hungry," "He is handsome," and "Chicago is far away" depend upon one's definition of "hungry," "handsome," and "far away." Consequently, such sentences are not statements, because they do not have a "truth value"—that is, because it is not possible to determine whether they are true or false. There are statements for which we do not know the truth value. For example, we do not know the truth value of Goldbach's conjecture, which states:

> "Every even integer greater than two can be written as the sum of two prime numbers."

Observe that $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, and $10 = 3 + 7 = 5 + 5$. To date, mathematicians have not been able to prove or disprove Goldbach's conjecture; however, it is a declarative sentence that is either true or false and not both true and false. Thus, Goldbach's conjecture is a statement or proposition in symbolic logic. Christian Goldbach made his famous conjecture in a letter written to Leonhard Euler on June 7, 1742. By October 2003, T. Oliveira e Salva had shown Goldbach's conjecture to be true for all even natural numbers less than 6×10^{16}.

In order to understand the statement of Goldbach's conjecture completely, you need to know the following definitions. The set of **natural numbers** is the set $\mathbf{N} = \{1, 2, 3, \ldots\}$. The natural numbers are also called the **counting numbers** and the **positive integers**. The basic theorems for the natural numbers appear in Section 2.1 and may be used in proofs which involve natural numbers. Let a and b be natural numbers. The number a **divides** b if there exists a natural number c such that $ac = b$. A **prime number** is a natural number greater than one which is divisible only by itself and one.

Example 1.1.1 Determine which of the following sentences is a statement.

a. How old are you?

b. $x + 3 = 5$

c. 2^{300} is a large number.

d. Help!

e. The author of this text was born in Washington, DC.

f. This sentence is false.

Solution

a. The sentence is a question (an interrogative sentence), and therefore it is not a statement.

b. The declarative sentence "$x + 3 = 5$" is true for $x = 2$ and false for all other values of x, so it is not a statement.

c. The declarative sentence "2^{300} is a large number" is not a statement, because the definition of "a large number" is not well-defined.

d. The sentence "Help!" is exclamatory, and hence it is not a statement.

e. The declarative sentence "The author of this text was born in Washington, DC" is true or false, but not both true and false, so it is a statement, even though few people would know whether the statement is true or false.

f. The declarative sentence "This sentence is false" is an interesting sentence. If we assign the truth value "true" or the truth value "false" to this sentence, we have a contradiction. Hence the sentence is not a statement. Because the sentence "This sentence is false" is neither "true" nor "false," it is called a **paradox**. ■

All statements can be divided into two types—simple and compound. A **simple statement** (**simple proposition**) is a statement which does not contain any other statement as a component part. Every **compound statement** (**compound proposition**) is a statement that does contain another statement as a component part. Every statement we have examined thus far is a simple statement. Compound statements are formed from simple statements using the logical connectives "and," "or," and "not."

Let P denote a statement. The **negation** of P, denoted by $\neg$ P, is the statement "not P." The negation of P is false when P is true, and the negation of P is true when P is false. For example, the negation of the statement "Five is a prime" is the statement "Five is not a prime." And the negation of the statement "Six is an odd number" is the statement "Six is not an odd number." In English, it is also possible to indicate the negation of a statement by prefixing the statement with the phrase "it is not the case that," "it is false that," or "it is not true that." For instance, the statement "It is not true that gold is heavier than lead" is true.

The **conjunction** of two statements P, Q, denoted by P $\wedge$ Q, is the statement "P and Q." The conjunction of P and Q is true if and only if both P and Q are true. Let M be the statement "It is Monday" and let R be the statement "It is raining." The statement M $\wedge$ R is "It is Monday, and it is raining." In English, several other words such as "but," "yet," "also," "still," "although," "however," "moreover," "nonetheless," and others, as well as the comma and semicolon, can mean "and" in their conjunctive sense. For instance, the statement "It is Monday; moreover, it is raining" should be translated to symbolic logic as M $\wedge$ R.

The **disjunction** of two statements P, Q, denoted by P ∨ Q, is the statement "P or Q." The disjunction of P and Q is true if P is true, if Q is true, or if both P and Q are true. In English, the word "or" has two related but distinguishable meanings. The "or" appearing in the definition of disjunction is the **inclusive or**. The **inclusive or** means "one or the other or both." In legal documents, the meaning of the inclusive "or" is often made more explicit by using the phrase "and/or." For example, the statement "This contract may be signed by John and/or Mary" means the contract is legally binding when signed by John, by Mary, or by both. On the other hand, the **exclusive or** means "one or the other but not both." For example, the statement "Ann will marry Ben or Ann will marry Ted" means either Ann will marry Ben or Ann will marry Ted but not both. In Latin there are two different words for the word "or." The word *vel* denotes the inclusive or, while the word *aut* denotes the exclusive or.

In the following two examples, we show how to write English statements in symbolic form and how to write symbolic statements in English.

Example 1.1.2 Write the following statements in symbolic form using ¬, ∧, and ∨.

a. Madrid is the capital of Spain and Paris is the capital of France.

b. Rome is the capital of Italy or London is the capital of England.

c. Rome is the capital of Italy, but London is not the capital of England.

d. Madrid is not the capital of Spain or Paris is not the capital of France.

e. Paris is the capital of France, but London is not the capital of England or Madrid is the capital of Spain.

Solution

Let M stand for the statement "Madrid is the capital of Spain."

Let P stand for the statement "Paris is the capital of France."

Let R stand for the statement "Rome is the capital of Italy."

Let L stand for the statement "London is the capital of England."

a. The statement may be written in symbolic form as $M \wedge P$.

b. The statement written in symbolic form is $R \vee L$.

c. In symbolic form, the statement may be written as $R \wedge (\neg L)$.

d. The statement in symbolic form is $(\neg M) \vee (\neg P)$.

e. In symbolic form, the statement is $P \wedge ((\neg L) \vee M)$. ■

Example 1.1.3 Let C be the statement "Today the sky is clear."

Let R be the statement "It did rain."

Let S be the statement "It did snow."

Let Y be the statement "Yesterday it was cloudy."