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Wolfgang Eichhorn
Winfried Gleißner

Mathematics and Methodology for Economics

Applications, Problems and Solutions

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Preface

This book about mathematics and methodology for economics is the result of the lifelong teaching experience of the authors. It is written for university students as well as for students of a university of applied sciences. It is completely self-contained and does not assume any previous knowledge of high school mathematics. At the end of all chapters and sections, there are exercises such that the reader can test how familiar she or he is with the material of the preceding stuff. After each set of exercises, the answers are given to encourage the reader to tackle the problems.

The idea to write such a book was born in 1990 during an international meeting on functional equations which took place at the University of Graz, Austria. At this meeting a lot of fascinating applications of functional equations to solve mathematically formulated economic problems inspired János Aczél, Distinguished Professor of Mathematics, University of Waterloo, Ontario, Canada: He proposed to one of us (W.E.) to start such an adventure in a form of a textbook for beginners. Since then he supported the tentative steps into this direction by a great wealth of brilliant scientific advices. Later on he became for both of us the lodestar for our endeavour. Dear János, we owe you a great debt of gratitude.

For a basic course Chaps. 1 (sets, vectors, trigonometric functions, complex numbers), 3 (mappings and functions), 4 (vectors, matrices, systems of linear equations), 6 (functions, limits, derivations), 7 (important nonlinear functions), and 10 (integration) are sufficient. If a later course will discuss discrete models of economics, Chap. 12 (difference equations) should be covered, too. For continuous models, Chap. 11 (differential equations) is necessary. (However, we decided not to go very far into details.)

Chapter 2 gives an introduction to linear optimisation and game theory using production systems. These ideas are continued in Chaps. 5 and 9, which discusses the notion of a Nash Equilibrium. Chapter 8 deals with nonlinear optimisation.

Chapter 13, as the conclusion, reflects methodologically most of all that what we optimistically offered in Chaps. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

Many thanks go to Thomas Schlink for typing most of the manuscript in LATEX very conscientiously and to Dr. Roland Peyrer for his inspiring drawings, which were transformed to PSTricks, an additional package for graphics in Latex.

Karlsruhe, Germany
Landshut, Germany
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