The theory of generalised functions

D.S.JONES

The theory of generalised functions

D.S.JONES

Ivory Professor of Mathematics, University of Dundee

CAMBRIDGE UNIVERSITY PRESS

Cambridge

London New York New Rochelle

Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia

This book is a substantially revised and completely reset edition of
Generalised Functions, published by McGraw-Hill 1966
First published by Cambridge University Press 1982

Printed in Great Britain at the University Press, Cambridge

British Library cataloguing in publication data

Jones, Douglas Samuel
The theory of generalised functions.–2nd ed.
1. Distributions, Theory of (Functional analysis)
I. Title II. Generalised functions
515.7′82 QA324 80–41830

ISBN 0 521 23723 8

The theory of generalised functions

To Katie, Kim and Corrie
for the pleasure they have given
Ivy and myself

Preface

For some years I have been offering lectures on generalised functions to undergraduate and postgraduate students. The undergraduate course was based originally on M.J. Lighthill's stimulating book *An Introduction to Fourier Analysis and Generalised Functions* which contains a simplified version of a theory evolved by G. Temple to make generalised functions more readily accessible and intelligible to students. It is an approach to the theory of generalised functions which permits early introduction in student courses while retaining the power and practical utility of the methods. At the same time it can be developed so as to include the more advanced aspects appropriate to postgraduate instruction. This book has grown from the courses which I have given expounding the ramifications of the Lighthill–Temple theory to various groups of students. It is arranged so that sections can be chosen relevant to any level of course.

Much of the material was originally contained in my book *Generalised Functions*, published by McGraw-Hill in 1966, but this book differs from the earlier version in several major respects. The treatment and definitions of the special generalised functions which are powers of the single variable x have been completely changed as well as those of the powers of the radial distance in higher dimensions. A different definition of δ-functions, whose support is on a surface, has also been introduced. The properties of the hyperbolic and ultrahyperbolic distances have been tackled in another way, with consequences for the general quadratic form. Numerous subsequent formulae are thereby altered. Further, a section has been added on the Fourier transform of weak functions and ultradistributions.

The purpose of the first chapter is to summarise some of the basic theorems of analysis which are required in subsequent chapters. It is anticipated that most readers will have met this material in one form or another before reading this book. For this reason explanation and argument have been cut to a minimum and, consequently, this chapter is not a suitable first reading for those who have not met several of the analytical ideas before. Since the chapter is self-contained some readers will, I hope, find it a useful introduction to the notions and terminology employed in other books where the approach to the subject has a more topological character. Many readers will find it profitable, on a first reading, to start at Chapter 2 and read onwards, referring back to Chapter 1 only for notation and statements of theorems.

In Chapter 2 the properties of good functions are given. Generalised functions of a single variable are introduced in Chapter 3 via sequences of good functions. After an examination of the derivative, Fourier transform and limit, the general structure of a generalised function is determined.

Chapter 4 is concerned with some special generalised functions, their Fourier transforms and the evaluation of certain integrals which are too singular to be embraced by classical analysis. The final section contains a brief discussion of generalised functions on a half-line.

Chapter 5 is devoted to series of generalised functions and shows, in particular, that any generalised function can be represented as a series of Hermite functions. There is also a detailed investigation of expansions in Fourier series, many theorems being much simpler than in classical analysis.

The problem of multiplication and division is dealt with in Chapter 6; the properties of the convolution product are also derived.

Generalised functions of several variables are introduced in Chapter 7. Most of the results are obvious generalisations of those for a single variable but new features are the direct product and the Fourier transform with respect to one of several variables. The last sections deal with spherically symmetric generalised functions and integration with respect to a parameter.

Chapter 8 treats the difficult problem of changing variables in a generalised function. This leads naturally to δ-functions on a hypersurface and the meaning to be attached to powers of the hyperbolic distance and its generalisations.

The asymptotic evaluation of Fourier integrals and the method of stationary phase in several variables comprise Chapter 9.

Applications of generalised functions are considered in Chapter 10. Particular reference is made to integral equations, ordinary and partial differential equations, as well as correlation theory.

Chapter 11 brings in the notion of a weak function, which is not so restricted at infinity as the generalised function. The significance of weak functions in solving integral equations, ordinary differential equations and in the justification of the operational method is shown. The Fourier transform of a weak function and ultradistributions are discussed, as well as the relation between weak functions and distributions.

The Laplace transform of a weak function is defined in Chapter 12 and a number of applications is given.

Exercises are given at various stages throughout the chapters. Most of them are to enable the reader to become thoroughly familiar with the theory, though some are extensions of theorems in the text. There are also some exercises which are worded so that they could be used as topics for minor theses. It is hoped that this variety will provide instructors with plenty of flexibility.

The author takes this opportunity of expressing his thanks to Mrs D. Ross for turning his manuscript into legible typescript despite a certain obscurity about the way it was organised.

The author's gratitude to his wife Ivy, who manages to display nonchalance and good cheer whatever burden is imposed on her, is immeasurable.

University of Dundee D.S. Jones
October 1980

Contents

	Preface	xi
1	**Convergence**	**1**
1.1	Preliminary definitions	1
1.2	Sequences	3
1.3	Functions	5
1.4	Topological space	7
1.5	Compactness	9
1.6	Metric spaces	12
1.7	Function spaces	15
1.8	Numbers	19
1.9	Limits	21
1.10	Series	25
1.11	Derivatives	28
1.12	Integration	31
2	**Good functions**	**44**
2.1	Good functions	44
2.2	Fourier transforms	46
3	**Generalised functions**	**53**
3.1	Generalised functions	53
3.2	Conventional functions as generalised functions	58
3.3	The derivative	63
3.4	The Fourier transform	71
3.5	Limits	75
3.6	Classification of generalised functions	81
4	**Powers of x**	**94**
4.1	Change of notation	94
4.2	Powers of x	94
4.3	Even and odd generalised functions	97
4.4	Singular integrals	99
4.5	Fourier transforms	109
4.6	Generalised functions containing a logarithm	115
4.7	Integration	121

5	**Series**	**128**
5.1	General properties	128
5.2	Hermite polynomials	132
5.3	The expansion of a generalised function in Hermite polynomials	137
5.4	Functions of the class L_2	141
5.5	Fourier series	144
5.6	Periodic generalised functions	149
5.7	Poisson summation formula	155
5.8	An alternative definition of a periodic generalised function	160
6	**Multiplication and the convolution product**	**162**
6.1	Multiplication	162
6.2	Division	172
6.3	The convolution	176
6.4	Properties of the convolution	180
6.5	A generalisation of the convolution	183
6.6	The convolution for the class K_+	200
7	**Several variables**	**210**
7.1	Good functions	211
7.2	Generalised functions	216
7.3	The derivative	220
7.4	The Fourier transform	224
7.5	The general form of a generalised function	228
7.6	The direct product	235
7.7	Some special generalised functions	242
7.8	Multiplication	257
7.9	Integration with respect to a parameter	263
8	**Change of variables and related topics**	**267**
8.1	Rotation of axes	267
8.2	Change of variable: one-dimensional case	273
8.3	Change of variables: several dimensions	287
8.4	Delta functions	292
8.5	Periodic and other changes of variables	300
8.6	Singular integrals	308
8.7	The hyperbolic and ultrahyperbolic distances	310
8.8	Fourier transforms of the hyperbolic and ultrahyperbolic distances	329
8.9	The general quadratic form	337
8.10	The class $\boldsymbol{K}_+$	340

9	**Asymptotic behaviour of Fourier integrals**	**345**
9.1	The Riemann–Lebesgue lemma	345
9.2	Generalised functions with a finite number of singularities in one dimension	349
9.3	The method of stationary phase	357
9.4	Generalised functions with isolated singularities	362
9.5	Integrals over a finite domain	364
9.6	Stationary phase in higher dimensions	379
9.7	Inverse polynomials	386
10	**Some applications**	**390**
10.1	Integral equations	390
10.2	Hilbert transforms	396
10.3	Ordinary linear differential equations	405
10.4	Linear partial differential equations	410
10.5	Approximate behaviour of elementary solutions	427
10.6	Covariance and generalised functions of positive type	429
11	**Weak functions**	**434**
11.1	Weak functions in one dimension	434
11.2	The weak derivative and the weak limit	437
11.3	The classification of weak functions	439
11.4	Sundry results	444
11.5	Multiplication	447
11.6	The convolution	451
11.7	Volterra integral equations	458
11.8	Ordinary linear differential equations	460
11.9	Linear differential equations with constant coefficients	462
11.10	The operational method	467
11.11	Weak functions of several variables	478
11.12	Integration with respect to a parameter	484
11.13	Change of variable: single variable case	485
11.14	Change of variables: several variables	487
11.15	Ultradistributions and Fourier transforms	488
11.16	The relation between weak functions and distributions	495
	Appendix: Titchmarsh's theorem	497
12	**The Laplace transform**	**500**
12.1	The Laplace transform	500
12.2	Ordinary differential equations	506

12.3 Integral equations 509
12.4 Laplace transform of a weak function of several variables 510
12.5 Partial differential equations 513
12.6 The bilateral Laplace transform 516
12.7 Integral equations 519
12.8 The Wiener–Hopf integral equation 520
12.9 A related partial differential equation 524
Table of Fourier transforms **529**
Table of Laplace transforms **533**
Index **534**

1
Convergence

This chapter is concerned primarily with deriving certain theorems on convergence which will be required in subsequent chapters. It is expected that most readers will be familiar with the notions involved so that much of the material is given in a condensed manner. However, an attempt has been made to make the chapter self-contained. Some readers may find the chapter a helpful introduction to the ideas and terminology employed in other books on generalised functions. The reader who does not have a good background in analysis is strongly advised to go straight to Chapter 2 and to just refer to Chapter 1 for the theorems that are needed.

1.1. Preliminary definitions

A *set* is a collection of elements. A set containing no elements is called a *null* or *empty set*. There is no restriction on what an element is: it may be a number or a point or a vector and so on. Usually we shall call the elements points and take all sets to be sets of points in a fixed non-empty set Ω, which will be called a *space*. The empty set will be denoted by $\varnothing$ and the capitals $A, B, \ldots$ will denote sets. If ω is a point of A, we write $\omega \in A$; if ω is not a point of A, we write $\omega \notin A$. Another useful notation is $\{\omega | P\}$ for the set of points satisfying condition P ; for example, the set of points common to both sets A and B can be written $\{\omega | \omega \in A \text{ and } \omega \in B\}$.

A set of sets is called a *class*. The class of all sets in Ω is called the *space of sets* in Ω. A class of sets in Ω is a set in this space of sets so that all set theories apply to classes considered as sets in the corresponding space of sets. Classes will be denoted by the script capitals $\mathscr{A}, \mathscr{B}, \ldots$

If all the points of A are points of B we write $A \subset B$ or, equivalently, $B \supset A$. Obviously, $A \subset A$ and $\varnothing \subset A \subset \Omega$. If $A \subset B$ and $B \subset C$ then $A \subset C$. If $A \subset B$ and $B \subset A$ we write $A = B$.

The *intersection* $A \cap B$ is the set of all points common to A and B, i.e. if $\omega \in A$ and $\omega \in B$ then $\omega \in A \cap B$ and conversely. The *union* $A \cup B$ is the set of all points which belong to at least one of the sets A or B, i.e. if $\omega \in A$ or $\omega \in B$ then $\omega \in A \cup B$ and conversely. If $A \cap B = \varnothing$ the sets A and B are said to be *disjoint* and their union may then be called a *sum* and written as $A + B$, i.e. if $A \cap B = \varnothing$ then $A \cup B = A + B$.

The *difference* $A - B$ is the set of all points of A which are not in B, i.e. if $\omega \in A$ and $\omega \notin B$ then $\omega \in A - B$ and conversely. The difference $\Omega - A$ is called the *complement* of A and denoted by A^c; it is the set of all points which do not belong to A.

The following *commutative, associative* and *distributive* laws are valid, i.e.

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A;$$

$$(A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cap C = A \cap (B \cap C);$$

$$(A \cup B) \cup C = (A \cap C) \cup (B \cap C),$$
$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C).$$

Relations between sets and their complements are:

$$\Omega^c = \varnothing, \qquad \varnothing^c = \Omega, \qquad A \cap A^c = \varnothing, \qquad A + A^c = \Omega;$$

$$A - B = A \cap B^c, \qquad (A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c;$$

if $A \subset B$ then $A^c \supset B^c$.

The operations of union and intersection can be extended to arbitrary classes. Let I be a set, not necessarily in Ω, and corresponding to each $i \in I$ choose a set $A_i \subset \Omega$. The class of sets so chosen will be denoted by $\{A_i | i \in I\}$. For obvious reasons I is called an *index set*. The *intersection* of $\{A_i | i \in I\}$ is the set of all points which belong to every A_i and is denoted by $\bigcap_{i \in I} A_i$, i.e.

$$\bigcap_{i \in I} A_i = \{\omega | \omega \in A_i \text{ for every } i \in I\}.$$

The *union* $\bigcup_{i \in I} A_i$ is the set of all points which belong to at least one A_i, i.e.

$$\bigcup_{i \in I} A_i = \{\omega | \omega \in A_i \text{ for some } i \in I\}.$$

If $A_i \cap A_j = \varnothing$ for all $i, j \in I$, $i \neq j$, the class $\{A_i | i \in I\}$ is said to be a *disjoint class* and the union of its sets may be called a sum and denoted by $\sum A_i$.

If $\omega \notin A_i$ then $\omega \in A_i^c$ and conversely. Consequently

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c, \qquad \left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c. \tag{1}$$

By convention

$$\bigcup_{i \in \varnothing} A_i = \varnothing, \qquad \bigcap_{i \in \varnothing} A_i = \Omega. \tag{2}$$

It will be observed that the following *principle of duality* holds: *any relation between sets involving unions and intersections becomes a valid relation by replacing* $\cup, \cap, \varnothing, \Omega$ *by* $\cap, \cup, \Omega, \varnothing$ *respectively.*

Finally we introduce the notion of equivalence class. Suppose we have a rule R which places the sets A and B in one-to-one correspondence, which we denote by ARB. The relation is *reflexive*, ARA; *symmetric*, ARB implies BRA; *transitive*, ARB and BRC imply ARC. A reflexive, symmetric and transitive relation is called an *equivalence relation.* The class $\{B | BRA\}$ is called the *equivalence class* corresponding to A. In essence an equivalence class is determined by any one of its members.

A class or set is said to be *finite* if its elements can be put in one-to-one correspondence with the first n positive integers, for some n. It is said to be *denumerable* if it can be put in one-to-one correspondence with all the positive integers. It is said to be *countable* if it is either finite or denumerable.

1.2. Sequences

For each value of $n(=1, 2, \dots)$ take a corresponding set A_n. The ordered denumerable class $A_1, A_2, \dots$ is called a *sequence* and is denoted by $\{A_n\}$. It is not necessary that $A_m \neq A_n$. The *limit superior* $\overline{\lim}_n A_n$ is defined by

$$\overline{\lim_n} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n;$$

it consists of the set of all those points which belong to infinitely many A_n. The *limit inferior* $\underline{\lim}_n A_n$ is defined by

$$\underline{\lim}_n A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n;$$

it consists of the set of all those points which belong to all but a finite number of A_n. Every point which belongs to all but a finite number of A_n belongs to infinitely many A_n so that

$$\underline{\lim}\, A_n \subset \overline{\lim}\, A_n.$$

If $\underline{\lim}\, A_n \supset \overline{\lim}\, A_n$ then $\underline{\lim}\, A_n = \overline{\lim}\, A_n$ and if this common set be denoted by A the sequence $\{A_n\}$ is said to *converge* to A.

A sequence is said to be *non-decreasing* if $A_n \subset A_{n+1}$ for each n; *non-increasing* if $A_{n+1} \subset A_n$ for each n. A *monotone* sequence is one which is either non-decreasing or non-increasing. *Every monotone sequence converges and, if it is non-decreasing,* $\overline{\lim}\, A_n = \bigcup_{k=1}^{\infty} A_k$, *whereas if it is non-increasing,* $\overline{\lim}\, A_n = \bigcap_{k=1}^{\infty} A_k$. This follows at once from the definitions.

The idea of sequence occurs in other ways; thus the *sequence* $\{\omega_n\}$ is the ordered denumerable set of points $\omega_1, \omega_2, \ldots$ A *subsequence* is obtained by selecting a sequence $\{n_i\}$ of positive integers with $n_i > n_j$ when $i > j$ and selecting the terms ω_{n_i} of the original sequence; the result is a sequence $\{\omega_{n_i}\}$ whose ith term is the n_ith term of the original sequence.

Many sequences involve real numbers, whose properties we now briefly review. A set X of real numbers is *bounded above* by the real number b if $x \leq b$ for every $x \in X$; b is called an *upper bound* for X. If b is an upper bound for X, if c is any other upper bound and if $b \leq c$ whatever c is then b is the smallest possible upper bound; in that case b is known as the *least upper bound* or *supremum* of X and written sup X. Sometimes the notation l.u.b. X is used. By reversing the inequalities in these definitions we define *bounded below, lower bound, greatest lower bound* or *infimum* of X (written inf X).

A fundamental postulate is: *every non-empty set of real numbers which is bounded above possesses a real supremum.* If the non-empty set X of real numbers is bounded below, the set $\{-x | x \in X\}$ is bounded above and hence possesses a real supremum. Therefore X has a real infimum, i.e. a non-empty set bounded above and below possesses both a real supremum and a real infimum.

The supremum of a sequence $\{x_n\}$ is denoted by $\sup_n x_n$. The *limit superior* is defined by

$$\overline{\lim_n}\, x_n = \inf_k \sup_{n \geq k} x_n$$

and the *limit inferior* by

$$\underline{\lim_n}\, x_n = \sup_k \inf_{n \geq k} x_n .$$

If a sequence is bounded above and below, it possesses both a real supremum and a real infimum and so has both a limit superior and a limit inferior.

The ordinary number system consists of *finite numbers*; the *extended real number system* is obtained by adding the *infinite numbers* ∞ and $-\infty$. These symbols have the properties:

$$x + (\pm\infty) = (\pm\infty) + x = \pm\infty, \quad \frac{x}{\pm\infty} = 0 \quad \text{if} \quad -\infty < x < \infty;$$

$$x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty & \text{if } 0 < x \leq \infty \\ \mp\infty & \text{if } -\infty \leq x < 0. \end{cases}$$

The expression $\infty - \infty$ is meaningless so that if one of the sum of two numbers be $\pm\infty$ the other must not be $\mp\infty$ for the sum to exist.

Any set of extended real numbers has both a supremum (which may be infinite) and an infimum. Consequently *every sequence of extended real numbers has a limit superior and a limit inferior*. Moreover, if inclusion, union and intersection of numbers be identified with $x \leq y$, $\sup_{i \in I} x_i$, $\inf_{i \in I} x_i$ respectively these operations have the properties of the corresponding set operations; thus *monotone sequences of extended real numbers* (i.e. $x_{n+1} \geq x_n$ or $x_{n+1} \leq x_n$ for all n) *always converge (possibly to infinity)*.

The set of all finite numbers $-\infty < x < \infty$ is the *real line* R_1 or $(-\infty, \infty)$; the set $-\infty \leq x \leq \infty$ is the *extended real line* $\bar{R}_1$ or $[-\infty, \infty]$.

1.3. Functions

If a rule is provided which associates with each $\omega \in \Omega$ a point $\omega' \in \Omega'$ we say that a *function f on Ω* or a *function from Ω to Ω′* is defined. The space Ω is called the *domain* of f. The point ω' which