Peking University Series in Mathematics — Vol. 7

SELECTED PAPERS OF WEIYUE DING

Editor
You-De Wang

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This collection covers all papers and partial talks given by Prof Weiyue Ding, who was a member of the Chinese Academy of Sciences. Prof Weiyue Ding devoted his academic career to the research in the field of ordinary differential equations and geometric analysis, e.g. Poincaré–Birkhoff fixed point theorems, blow-up analysis for heat flow of harmonic maps.

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Preface

To commemorate the late Professor Weiyue Ding, a renowned mathematician and a member of the Chinese Academy of Sciences, in early 2015 Professor Kung-Ching Chang proposed to publish the entire collection of Professor Weiyue Ding's research works. This proposal received enthusiastic support from the School of Mathematical Sciences at Peking University and the Academy of Mathematics and Systems Science of the Chinese Academy of Sciences. It also received overwhelming support from Professor Weiyue Ding's friends and former students.

Many people have contributed to the publication of this collection. In particular, we would like to thank Professors Wenxiong Chen and Xuefeng Wang for helping me to write the biography of Professor Weiyue Ding, detailing his life as well as his research contributions. We wish to express our gratitude to Professors Daqian Li and Song Jiang for their care and help throughout the publication process. We also thank the editorial boards of the following journals: *Science China Mathematics*, *ACTA MATHEMATICA SINICA* (*Chinese Series and English Series*), *Chinese Annals of Mathematics*, *Advances in Mathematics* (*China*) and *Journal of Partial Differential Equations* for their assistance. Finally, we would like to thank Mr. Hongbing Shi and Ms. Chionh Eng Huay of World Scientific Publishing Co. Pte. Ltd. for their dedication and hard work.

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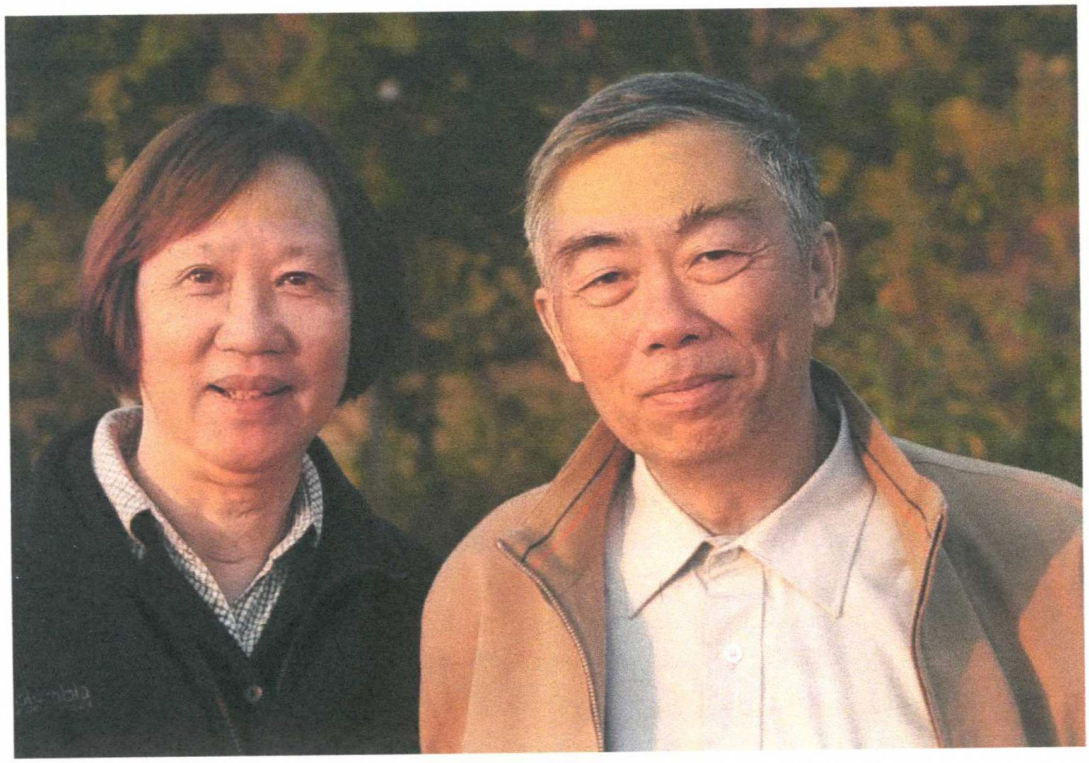

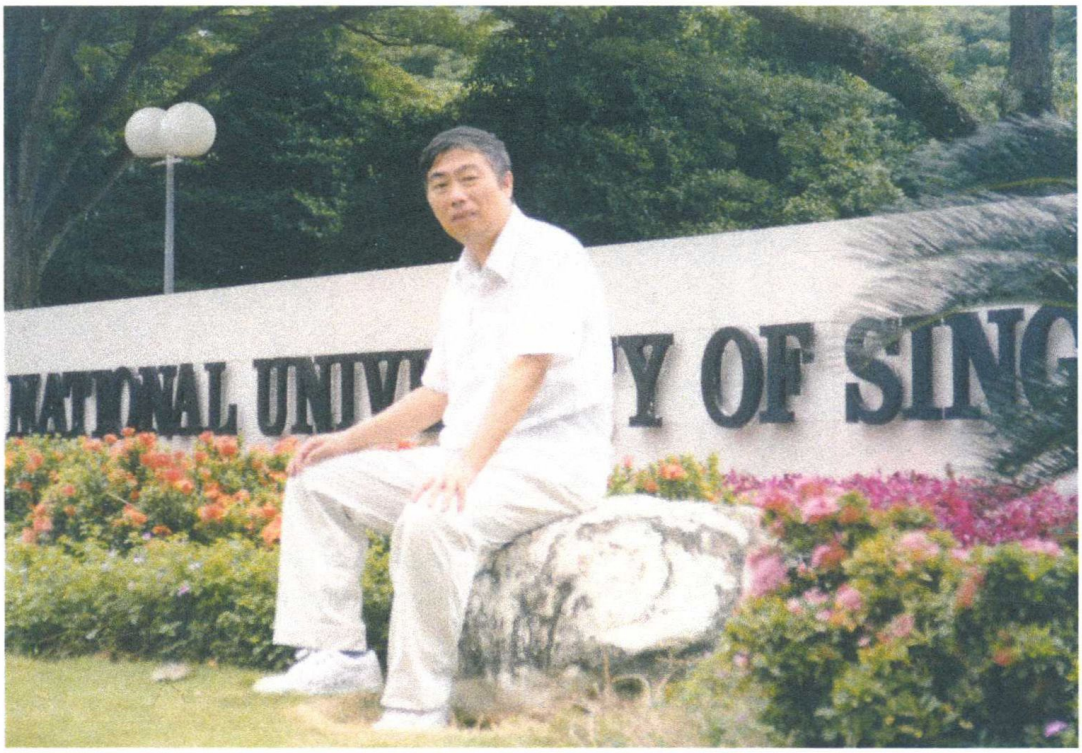

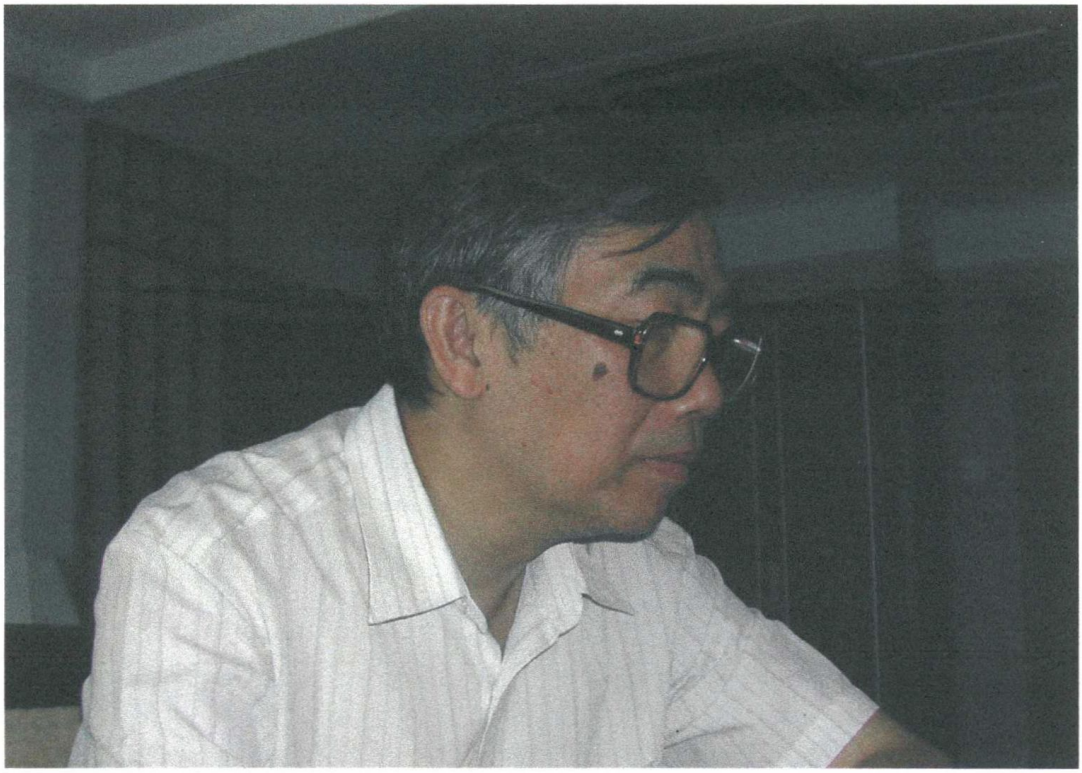

Weiyue Ding

Professor Weiyue Ding (1945–2014) was a second generation Chinese mathematician educated after the establishment of New China in 1949. He made original and significant contributions to research in geometric partial differential equations (PDE), including semi-linear elliptic PDE, harmonic maps and minimal surfaces, prescribing scalar curvature problem, Schrödinger flow, and the existence of Kähler–Einstein metrics. Due to these achievements, he received a National Natural Science Award and the prestigious S. S. Chern Award in Mathematics in 1993; he was elected as a member of Chinese Academy of Sciences in 1997; he was invited to give a 45-minute plenary lecture at the International Congress of Mathematicians (ICM) in 2002.

Professor Ding worked tirelessly to realize the dream of several generations of Chinese mathematicians: making China a strong nation in mathematics. He was an inspiring teacher, nurturing students who are now leading experts in their research fields. He served as the vice president of the Chinese Mathematical Society, and organized the successful ICM held in 2002 in Beijing.

I. Academic Career

Weiyue Ding's hometown is Zhou Shan, Zhejiang Province. He was born in Shanghai on April 26, 1945. He then moved to Beijing with his parents in 1951. His father was an architect, and his mother a housewife. In Beijing, he attended elementary school for five and a half years, and then the prestigious Beijing Fourth Middle and High School for six years. He was admitted to Peking University, Department of Mathematics and Mechanics in 1962. During his sophomore year, he chose his major to be in mathematics. The five years' undergraduate program (standard length at that time) was interrupted by the Cultural Revolution (1966–1976), which not only cut the

normal study period to three years, but also delayed the graduation year to 1968. At that time, the Party Central Committee asked intellectuals to be "re-educated", so in September 1968, Ding and his peers were dispatched to an army reclamation farm in Anhui Province to do hard labor. In the farm, he met Miss Zheyu Zhu, who became his wife later. Starting from June 1970, as a chemical plant worker he first worked in Qinghai Province until 1971, then worked in Sichuan Province until the fall of 1978, which was the year of great turning point in Weiyue Ding's life.

In that year, China restarted to recruit graduate students through examinations. Ding applied for the Master's program at Institute of Mathematics, Chinese Academy of Sciences (CAS). There were at least 1,300 applicants, but the Institute admitted only 30 students, including Ding. It was said that he was ranked Number 1 in the Ordinary Differential Equations Exam, and second in the Partial Differential Equations Exam. Under the supervision of Professor Guangyin Wang, he obtained his Master's degree in 1981, and then worked as a researcher at the same institute ever since.

In 1986, CAS awarded Weiyue Ding his Ph.D. degree and promoted him to Associate Professor; and in 1988, to Full Professor. In 2000, he started to work at School of Mathematics, Peking University, while keeping his position and title at CAS. In May the same year, he was appointed as the chair of the Institute of Mathematics at Peking University.

II. Academic Achievements

At the beginning of his research career, Weiyue Ding was interested in the existence of periodic solutions of ODEs. In September 1982, he went abroad for the first time, visiting University of Minnesota, where he met Professor Wei-Ming Ni, an expert in PDEs. Although he just got exposed to partial differential equations, he had read many papers on ODEs which involved methods of nonlinear functional analysis. He and Wei-Ming Ni had quite close interests in research, and they began working on some semilinear elliptic equations on the whole space. This was the time when Weiyue Ding's research interest turned to PDEs.

From the end of '70s to the beginning of '80s, Differential Geometry entered its golden age, in which, several important problems were solved and some new theories and methods emerged. In 1984, the world famous mathematician Shing-Tung Yau came to give lectures in China, and he encouraged Chinese mathematicians to learn and study Geometric Analysis. As a result, the joint Nonlinear Analysis Group of Peking University and Institute of Mathematics of CAS gradually engaged in the study of Partial

Differential Equations in Differential Geometry. From that time on, Weiyue Ding turned his research attention from Partial Differential Equations to Geometric Analysis, specializing in harmonic maps between Riemannian manifolds and prescribing scalar curvature problems in Conformal Geometry.

II.1. *Poincaŕe–Birkhoff Fixed Point Theorem and Semilinear Elliptic Equations*

Most researchers in Ordinary Differential Equations are aware of the so-called Poincaré's Final Geometric Theorem, and nowadays, we usually call it Poincaré–Birkhoff Fixed Point Theorem. At the start of his research career, Ding generalized this result. This generalization turns out to be sharp, as demonstrated by two examples constructed by two mathematicians recently, who called his theorem as "*Ding's version of Poincaré–Birkhoff Theorem*". This theorem can be conveniently and very effectively applied to study periodic solutions for ODEs, and is of fundamental importance. It has been cited constantly up to the present day.

Jointly with Wei-Ming Ni, Weiyue Ding wrote two papers concerning the existence of positive solutions for semilinear equations on Euclidean spaces of dimensions greater than two, which were published in *Duke Math. J.* (1985) [5][a] and *Arch. Rational Mech. Anal.* (1986) [8]. They introduced Nehari manifold into the study of this kind of problems, and discovered the delicate "Ding–Ni phenomenon". Their work has generated a lot of activities by other researchers; the Nehari manifold technique has become a standard tool. These two papers marked the turning point of Ding's research from ODEs to PDEs.

In 1985, Weiyue Ding applied symmetric variational principle to obtain infinitely many finite energy, sign changing solutions for the Yamabe equation, which provided a counterexample to some people's conjecture on the nonexistence of such solutions.

Another problem of common interest at that time was whether a positive solution can be found for the Dirichlet problem of such an equation on bounded smooth domains. Pohozaev proved the nonexistence of such solutions if the domain is star-shaped. Later, researchers discovered that if the topology of the domain was nontrivial, the equation may possess a positive solution. Then a natural question was: "Does the topology of the domain determine the existence of positive solutions?" In 1988, Weiyue Ding constructed a family of shrinkable domains for which the above problems possess positive solutions. This result showed that it was the geometry, not

[a]The number indicated in the brackets represents the paper in the table of contents.

the topology, that determines the solvability of such problems. The paper was published in *J. Partial Differential Equations* (1989) [15], and has been widely cited by others.

II.2. *Prescribing Scalar Curvature and Moser–Trudinger Inequality*

Blow-up analysis is a powerful tool in studying variational problems where the lack of compactness occurs. Chang and Carleson once used this approach to obtain the existence of extremal function for a Moser–Trudinger inequality on unit spheres.

In 1985, Weiyue Ding and Wenxiong Chen studied the prescribing scalar curvature problem on 2-dimensional sphere, and for the first time, they obtained sufficient conditions for the existence of solutions in the case where the given functions were not symmetric. It became known later that Alice Chang and Paul Yang also obtained similar results independently, which were reported by the former at ICM at Berkeley in 1986.

Later, Weiyue Ding together with Jürgen Jost, Jiayu Li, and Guofang Wang, used a similar idea to investigate the existence of extremal functions for weak Moser–Trudinger inequality on compact surfaces. The variational equations are related to Gauss curvature equations and Chern–Simons–Higgs equations. They established a sufficient condition for the existence of solutions in the critical case. This was their first and most significant paper among a series of papers on this subject [31]. Their work has been cited widely.

II.3. *Kähler–Einstein Metrics*

The existence of Kähler–Einstein metrics on compact Kähler manifolds in complex dimension one is equivalent to the prescribing scalar curvature problem on 2-dimensional manifolds. Moser–Trudinger inequality plays an important role in the analysis of the latter.

Motivated by this, and by a paper of Aubin, Weiyue Ding introduced a functional, called "F-functional", on Kähler manifolds with positive first Chern-class, which he used to obtain sufficient conditions for the existence of Kähler–Einstein metrics. Later, in his joint article with Gang Tian, they pointed out that this "F-functional" could be defined on any Kähler classes. At the beginning of '90s, Weiyue Ding suggested to Gang Tian that one could study the nonexistence of Kähler–Einstein metrics through the discussion of lower unboundedness of Mabuchi's "K-energy". Based on this suggestion, Gang Tian introduced the concept of "generalized Futaki-invariant", and

provided some nonexistence examples. This new invariant was the starting point of Tian's later important theory on "K-stability".

Recently, Ding's early work in Kähler geometry has been revisited by many people working in Kähler geometry. Swedish mathematician Bo Berndtsson discovered that Ding's functional is geodesically convex in the space of Kähler metrics on Fano manifold, with respect to the Riemannian metric defined by Mabuchi–Semmes–Donaldson. This gives rise to an important generalization of Bando–Mabuchi uniqueness theorem to the case of singular Fano varieties.

On the algebro-geometric side, Ding's functional has led to the notion of Ding-stability (introduced by another Swedish mathematician Robert Berman), which is roughly speaking, the asymptotic derivative of Ding functional along a degeneration. This is closely related to the much more well-studied notion of K-stability. Ding-stability has been further investigated by a young Japanese mathematician Kento Fujita, to prove the deep result that complex projective spaces have maximum volume among all Fano Kähler–Einstein manifolds.

II.4. *Harmonic Maps and Minimal Surfaces*

In 1981, the publication of Sacks and Uhlenbeck's well-known paper on 2-dimensional harmonic maps and minimal spheres exerted a great impact on the development of Geometric Analysis. Weiyue Ding investigated, systematically and in-depth, many important problems in harmonic maps, and obtained a series of results of high value. For example, he beautifully solved the blowing up problem in the heat flow for harmonic maps and the existence of symmetric harmonic maps between spheres.

In 1985, Weiyue Ding completed his first paper on harmonic maps. He established a Lusternik–Schnirelmann theory for harmonic maps in two dimensions. As an application of this theory, he generalized a multiplicity result of Benci and Coron. The paper was published in *Acta. Math. Sinica* (1986) [10].

In 1975, Smith began his study on symmetric harmonic maps between spheres and obtained some existence results. Weiyue Ding employed the variational method to completely solve Smith's problem, providing a necessary and sufficient condition for the solvability. In particular, he used the *mountain pass lemma* to prove the necessary condition for the solvability, which is unorthodox (usually people use *mountain pass lemma* to prove sufficient conditions for existence). This paper was published in *Commun. Math. Phys.* (1988) [13]. Later, he and his co-authors investigated the Hopf structure

for harmonic maps between spheres and obtained necessary and sufficient conditions for the existence.

He and Youde Wang studied the existence of harmonic maps from non-compact complete manifolds to manifolds with non-positive curvature. Their method can be used to study the existence of solutions for other PDEs arising from Geometry, such as the existence of Hermitian–Einstein metrics for holomorphic vector bundles on some complete Kähler manifolds.

Since 1964, after Eells and Sampson proved the existence of global solutions of harmonic map heat flows from a compact manifold to a non-negatively curved compact manifold; however, whether or not such a flow could blow up in finite time in the general case had eluded researchers for many years. In 1988, under a family of mild conditions, Weiyue Ding proved that a harmonic map heat flow on a 3-dimensional compact manifold can blow up in finite time. This paper was published in *Advances in Mathematics (China)*. In December 1964, when visiting the International Center for Theoretical Physics (ICTP), Weiyue Ding completed an important paper with Yunmei Chen, a postdoctoral fellow there. In that paper, they generalized Ding's previous result on the finite time blow up of the harmonic map heat flow from dimension 3 to any dimensions. The key for their generalization is the monotonicity inequality for the solutions of the harmonic maps heat flow established by Struwe. The paper was published in the prestigious journal *Invent. Math.* (1990) [16].

Then the remaining problem was: "Can the 2-dimensional harmonic heat flow generate singularities in finite time?" This had drawn much attention from geometric analysts, and they had different conjectures.

In the summer of 1991, jointly with Gongqing Zhang and Rugang Ye, Weiyue Ding provided examples for the finite time blow up of 2-dimensional harmonic map heat flow. The key was a very delicate construction of sub-solutions. This answered the above question affirmatively. After the publication of this paper in *J. Diff. Geom.* (1992) [23], it was again another highly used paper.

The energy identity for the approximation sequences to the harmonic maps: In their well-known paper, Sacks and Uhlenbeck discovered for the first time that the approximation sequences they defined could produce blowing bubbles in the process of convergence. However, it was unclear whether there was an energy loss during the bubbling. This problem also arose in studying the blow up of 2-dimensional harmonic maps heat flows.

In 1994, Weiyue Ding and Gang Tian further investigated the convergent behavior of the "neck" during the bubbling process, and proved that there was no energy loss of the "neck" when the tension field of the approximation