ENERGY IN ELECTROMAGNETISM

H.G. BOOKER

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Preface

Is the teaching of electricity and magnetism in need of change? One might imagine that any change needed was accomplished with the development of the *Berkeley Physics Course* (McGraw Hill, New York, 1963–5) and with the appearance of *The Feynman Lectures in Physics* (Addison-Wesley, New York, 1964). Not so. Feynman himself says in his preface: 'In the second year I was not so satisfied. In the first part of the course, dealing with electricity and magnetism, I could not think of any really unique or different way of doing it – of any way that would be particularly more exciting than the usual way of presenting it. So I don't think I did very much in the lectures on electricity and magnetism.' The preface to the *Berkeley Physics Course* says: 'Our specific objectives were to introduce coherently into an elementary curriculum the ideas of special relativity, of quantum mechanics, and of statistical physics.' There is no mention of comprehensive rethinking of the presentation of electricity and magnetism.

The preface to Volume II of the *Berkeley Physics Course* dealing with electricity and magnetism says: 'The sequence of topics is, in rough outline, not unusual. . . . . The difference is most conspicuous in Chapters 5 and 6 where, building on the work of Vol. I, we treat electric and magnetic fields of moving charges as manifestations of relativity and the invariance of electric charge.' The book certainly does make it easier for a student to understand particle accelerators, but not ordinary electrical machines that do not involve relativistic velocities. I doubt if a student who has studied the book could explain in terms of simple physics not involving corrections of order $(v/c)^2$ how, for the same machine performing the same task, different charge distributions are acquired as seen by two different observers, one fixed relative to the field-winding and the other fixed relative is the armature-winding.

The preface to Volume II of the *Berkeley Physics Course* goes on to say: 'This approach focuses attention on some fundamental questions, such as: charge conservation, charge invariance, the meaning of field.' Yet the meaning of field is obscured in the book precisely where charge conservation is concerned. The Maxwell equations divide into two groups. There are two equations involving $\boldsymbol{E}$ and $\boldsymbol{B}$ that are directly concerned with the exercise of force in accordance with the expression $\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}$; they correspond to the kinetic equations in mechanics.

The other two Maxwell equations are relations between the quantities $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{J}$, ρ that are concerned with current and charge geometry; the unit of $\boldsymbol{H}$ is current per unit distance, that of $\boldsymbol{D}$ is charge per unit area, that of $\boldsymbol{J}$ is current per unit area, and that of ρ is charge per unit volume. These two Maxwell equations may be likened to the equations of kinematics in mechanics; they incorporate the principle of conservation of charge embodied in the equation of continuity. I know of no book that adequately brings out the quite different physical bases that exist for the $\boldsymbol{E}$, $\boldsymbol{B}$ Maxwell equations and for the $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{J}$, ρ Maxwell equations.

These considerations led me to question the conventional presentation of electricity and magnetism. In consequence, the entire basis of electricity and magnetism, as well as the methods used for teaching it, have been subjected to the kind of original thinking customarily reserved for research. In some areas this has confirmed the wisdom of the traditional approach, but in other areas improvements have come to light. In some cases, the fresh thought has actually resurrected in novel form ideas that people once discarded. But wherever the electric-flux-density vector $\boldsymbol{D}$ is concerned, the changes triggered by the fresh thinking have turned out to be substantial. Even in this respect, however, an interesting link exists to the early thinking of Faraday.

Radiation from antennas has always been treated by dissecting the current into elements and regarding each current element, together with the associated charges at its ends, as an electric dipole. It is possible, and frequently desirable, to treat any flow of free electric current as a distribution of free electric moment per unit volume $\boldsymbol{P}_f$. As in the case of bound current and charge associated with bound electric moment per unit volume $\boldsymbol{P}_b$, the free current density is the time-derivative of $\boldsymbol{P}_f$ and the free charge density is the negative divergence of $\boldsymbol{P}_f$. Propagation of electromagnetic waves in a plasma has been treated for half a century by handling the free electric moment per unit volume $\boldsymbol{P}_f$ in a plasma in the same way as is done for the bound electric moment per unit volume $\boldsymbol{P}_b$ in a dielectric. This means that the electric-flux-density vector used in connection with the complex dielectric constant (or tensor) of a plasma is not identical with the vector $\boldsymbol{D}$ introduced in most elementary textbooks in connection with dielectric insulators.

Care is necessary to distinguish between three different versions of the electric-flux-density vector in materials. For an oscillatory field of angular frequency ω in material of dielectric constant ϵ/ϵ_0 and conductivity σ, the three versions of the complex electric-flux-density vector at a point where the complex electric vector is $\boldsymbol{E}$ are $\epsilon_0\boldsymbol{E}$, $\epsilon\boldsymbol{E}$, and $(\epsilon - j\sigma/\omega)\boldsymbol{E}$. The first of these is the electric-flux-density vector to be used if one simply regards the material as creating an additional distribution of current and charge in free space. On the other hand, $\epsilon\boldsymbol{E}$ is an electric-flux-density vector that suppresses in Maxwell's equations the currents and charges associated with the bound electric moment per unit volume $\boldsymbol{P}_b$; this is convenient when discussing the design and operation of equipment where special attention needs to be paid to the free currents and charges on good conductors. But $(\epsilon - j\sigma/\omega)\boldsymbol{E}$ is an electric-flux-density vector that suppresses in Maxwell's equations the currents and charges associated with both the bound electric moment per unit volume P_b and the free electric moment per unit volume P_f; it is usually the convenient one to use when the wave character of the electromagnetic field under study is dominant. For any time-varying electromagnetic field in any material, including non-linear and non-isotropic materials, the three versions of the electric-flux-density vector are, respectively, $\epsilon_0 E$, $\epsilon_0 E + P_b$, and $\epsilon_0 E + P_b + P_f$.

The upshot of the thinking outlined in this preface has been a presentation of electricity and magnetism in three parts. These do not correspond to electrostatics, magnetostatics, and electromagnetism. Instead they deal with:

(1) electromagnetic fields in free space,
(2) electromagnetic fields in materials, and
(3) energy in electromagnetism.

Consistent with not requiring the reader to take leaps in the dark, Part 1 follows a direct route to the calculation of the electromagnetic field in free space caused by any known distribution of time-varying electric current and its associated distribution of time-varying electric charge. Dielectric and magnetic materials do not appear explicitly in Part 1; conductors appear only as locations for electric currents and charges.

Materials constitute, from an electromagnetic standpoint, distributions of current and charge in free space additional to the primary sources of the field. But usually these additional currents and charges are not immediately known. The local dipole moments are not known because they depend on the local electromagnetic field, which is just what we are trying to calculate. This is the fundamental complication created in electromagnetic theory by the presence of materials. It is this complication that is the subject of Part 2.

The magnetic vector potential at a point in an electromagnetic field is presented as the electromagnetic momentum per unit charge of a test charge located at the point, the total momentum of a particle being the vector sum of its mechanical momentum and its electromagnetic momentum. Currents round circuits are then seen as electronic fly-wheels for which the effect of charge is normally much greater than that of mass. Also, in Part 3, the reader is encouraged to think of the Poynting vector as representing a flow of energy in space that can be intercepted and measured in substantially the way in which flow of energy along a power line can be intercepted and measured. The vector field $E \times H$ is not then a by-product of Poynting's theorem plagued by an indeterminacy involving closed flow of energy. On the contrary, the Poynting vector is something as vivid as the product VI on the basis of which consumers of electric power pay their bills.

In the division of subject matter described above, the present volume constitutes Part 3. Chapters 1 and 2 summarise the contents of Parts 1 and 2 concerning electromagnetic fields in free space and in materials. They therefore deal in outline form with a number of the approaches developed in Parts 1 and 2. A few problems are provided, not as exercises, but to incorporate in suscinct form relevant additional material.

A number of the ideas employed in this text have been under development for a considerable period of time. Some originated in the Bell Telephone Laboratories before World War II, some came to the fore in connection with radar research, some appeared in lectures given by the author at the University of Cambridge, and some in a text (Henry G. Booker, An Approach to Electrical Science, McGraw Hill, New York, 1959) written at Cornell University. But it is a sequence of courses presented in recent years at the University of California, San Diego, that has provided the opportunity to assemble this book. The work was supported by an Instructional Improvement Grant from the University of California. My thanks are due to Pat Norvell for converting almost illegible manuscript into readable typescript, to Jerry Ferguson and Reza MajidiAhi for performing the computations needed for some of the diagrams, and to Hari Vats for reading the proof.

Henry G. Booker
January 1981

Contents

Chapter 1

Electromagnetic fields in free space

1.1 Introduction

Before we can study energy in electromagnetism, it is necessary to understand electromagnetic fields. In this chapter, we shall summarise the facts concerning the electromagnetic fields created by known time-varying distributions of electric charge and current in otherwise free space.

Material consists of an aggregate of particles in space many of which are charged and are in motion. Such a distribution of moving charged particles constitutes electric charge and current in space. However, the electric charge and current densities in a material near a point 0 depend on the electromagnetic field in the neighbourhood of 0, that is, on the very electromagnetic field that we seek to calculate. This causes a complication that we shall not address until the following chapter.

Generally, we shall suppose in this chapter that the distributions of electric charge and current exist in otherwise free space, and that they are known functions of position and time. The problem is to calculate the electromagnetic field that they create. However, it will sometimes be instructive to regard the electromagnetic field as known in space and time, and to enquire what distributions of electric charge and current are needed to create it.

1.2 Force in an electromagnetic field

The exercise of force in an electromagnetic field is described by means of the electric field and the field of magnetic flux density. At a point in an electromagnetic field where the electric vector is $\boldsymbol{E}$ and the magnetic-flux-density vector is $\boldsymbol{B}$, the force per unit charge that would act on a test charge at the point is

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \tag{1.1}$$

where $\boldsymbol{v}$ is the velocity of the test charge relative to the observer. Different observers in motion relative to each other observe different values of $\boldsymbol{v}$, but they also observe different values of $\boldsymbol{E}$ and $\boldsymbol{B}$. When all observers have evaluated $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$

they are in agreement with each other on the force exerted. This agreement constitutes equality when the velocities are small compared with the velocity c of light in free space.

The fields described by the electric vector E and the magnetic-flux-density vector B are related by Faraday's law of induction. If ds is a vector element of length of any closed curve C in any electromagnetic field, and if E is the electric vector at ds, then (see Fig. 1.1)

$$\int_C \boldsymbol{E}\cdot d\boldsymbol{s} \tag{1.2}$$

is the circulation of E round C in the direction of ds. If S is a surface spanning C and dS is a vector element of area of S at a point where the magnetic-flux-density vector is B, then

$$\int_S \boldsymbol{B}\cdot d\boldsymbol{S} \tag{1.3}$$

is the magnetic flux crossing S in the direction of dS. In terms of these concepts, Faraday's law of induction states that, for any closed curve C fixed relative to the observer and spanned by a surface S,

$$\int_C \boldsymbol{E}\cdot d\boldsymbol{s} = -\frac{\partial}{\partial t}\int_S \boldsymbol{B}\cdot d\boldsymbol{S} \tag{1.4}$$

where t denotes time. Faraday's law of induction expresses the fact that the circulation of the electric vector round any fixed closed curve C in any electromagnetic field is equal to the time-rate of decrease of the magnetic flux crossing a surface S spanning C. The law implies the adoption of a sign convention, and we employ the right-hand-screw rule. We regard the vector product in expr. 1.1 as defined with the aid of the right-hand-screw rule, and we also regard the direction of the vector element of area dS of S in Fig. 1.1 as related by the right-hand-screw rule to the direction of the vector element of length ds of the rim C.

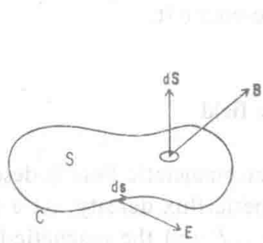

Fig. 1.1 *Illustrating calculation of the circulation of the electric vector round a closed curve C and calculation of the magnetic flux crossing a spanning surface S*

Eqn. 1.4 applies in particular to the rim of every fixed element of area dS in space, no matter what its location or what its orientation. The curl of a vector field E (see Appendix A) is a derived vector field such that, for any vector element of area dS at a location where the derived vector field is curl E, the right-hand related circulation of E round the rim of dS is (curl E)·dS. Hence, by definition of curl, eqn. 1.4 implies that, at all locations at all times,

$$\text{curl}\,E = -\frac{\partial B}{\partial t} \tag{1.5}$$

This is Maxwell's electric curl equation. It is a relation between the two vector fields (E, B) involved in describing how force is exerted in an electromagnetic field. It is true in every electromagnetic field at every point of space where the derivatives involved in the equation exist.

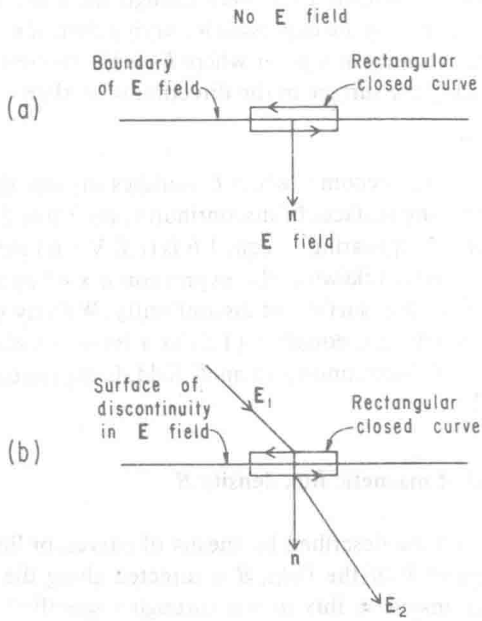

Fig. 1.2 *Illustrating vanishing of the surface curl of E*
a At a fixed boundary of an E field
b At a fixed surface of discontinuity in an E field
Note that the rectangular loop, the vector E_1 and the vector E_2 are not in general coplanar

At a boundary of an E field, the space derivatives of E involved in eqn. 1.5 do not in general exist. However, at a boundary fixed relative to the observer, we can apply the integral version of eqn. 1.5, given in eqn. 1.4, to the perimeter of a narrow rectangular element of area for which the long edges are on opposite sides of the boundary (see Fig. 1.2*a*). As the length of the narrow edges of the rectangle tends to zero, the contribution from the right-hand side of eqn. 1.4 also tends to zero. Moreover, the circulation of E round the rim of the rectangle reduces to the

the product of one of the long edges and the component of E along it. Since all orientations of this edge in the boundary are possible, we deduce that the tangential component of E must always vanish at a fixed boundary of an electric field. This may be expressed by saying that, if n is a unit normal to the boundary directed into the field at a point where the adjacent electric vector is E, then (see Fig. 1.2*a*)

$$n \times E = 0 \tag{1.6}$$

The boundary of an electric field is a special case of a surface of discontinuity. At any fixed surface of discontinuity of the E field we may apply eqn. 1.4 to the perimeter of a narrow rectangular element of area for which the long edges are on opposite sides of the surface (see Fig. 1.2*b*). We deduce that there is no discontinuity in the tangential component of E even though there is a discontinuity in the normal component. This may be expressed by saying that, if n is a unit normal to a fixed surface of discontinuity at a point where $E|$ is the discontinuous increase in E experienced on crossing the surface in the direction of n, then

$$n \times E| = 0 \tag{1.7}$$

Eqn. 1.6 is what eqn. 1.7 becomes when E vanishes on one side of the surface of discontinuity. For moving surfaces of discontinuity, see Table 2.4.

The expression $n \times E$ appearing in eqn. 1.6 is (c.f. $\nabla \times E$) the surface curl of E at the boundary of the field. Likewise, the expression $n \times E|$ appearing in eqn. 1.7 is the surface curl of E at any surface of discontinuity. We may say that the counterpart of Maxwell's electric curl equation (1.5) at a fixed boundary of an E field, or at any fixed surface of discontinuity in an E field, is expressed by the vanishing of the surface curl of E.

1.3 The vector field of magnetic flux density *B*

The vector field B may be described by means of curves, or lines, of magnetic flux such that, at each point P of the field, B is directed along the line through P. The aggregate of lines of magnetic flux drawn through a specified closed curve constitutes a tube of magnetic flux. The entire magnetic field may by analysed into thin tubes of magnetic flux.

A thin tube of magnetic flux has the property that, if dS is the vector area of a cross-section of the tube (not necessarily a normal cross-section) at a point where the magnetic-flux-density vector is B then, at each instant of time, the quantity

$$B \cdot dS \tag{1.8}$$

has the same value for all locations along the tube and for all cross-sections of the tube. The quantity (1.8) is known as the strength of the tube. Tubes of magnetic flux have the property that they have no beginning and no end. They frequently take the form of closed tubes. Tubes of magnetic flux are therefore endless tubes of constant strength.

Let dS be a vector element of area of an unclosed surface S, and let B be the magnetic-flux-density vector at dS at time t. Then, in accordance with expr. 1.8, the tube of magnetic flux formed by lines of magnetic flux through the rim of dS at time t has strength $B \cdot dS$, and the sum of the strengths of the tubes crossing S at time t is

$$\int_S B \cdot dS. \tag{1.9}$$

This is the magnetic flux crossing S at time t in the direction defined by dS.

If C is the rim of the surface S, the integral (1.9) is also described as the magnetic flux threading C at time t. Because $B \cdot dS$ does not vary along a tube of magnetic flux, the sum of the strengths of the tubes threading a closed curve C may be calculated from expr. 1.9 using any surface S spanning C. The magnetic flux threading a closed curve C is independent of the spanning surface S used in the calculation. This property of the B field is relevant to Faraday's law of induction. The value of the surface integral on the right-hand side of eqn. 1.4 is independent of the surface S used to span the closed curve C round which the circulation of E is evaluated.

The fact that each thin tube of magnetic flux not only has the same value of $B \cdot dS$ at all points, but is also endless, has the following consequence. For any surface Σ enclosing a volume V in space, each tube that conveys magnetic flux into V across Σ also conveys the same amount of flux across Σ out of V (see Fig. 1.3). Hence, for any closed surface Σ in any electromagnetic field,

$$\int_\Sigma B \cdot dS = 0 \tag{1.10}$$

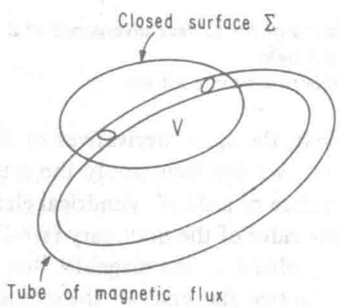

Fig. 1.3 *Illustrating the vanishing of the magnetic flux out of a closed surface* Σ

Eqn. 1.10 applies in particular to the surfaces of every element of volume in space. The divergence of a vector field B (see Appendix A) is a derived scalar field such that, for any element of volume $d\tau$ at a location where the derived scalar field is div B, the flux of B out of the element of volume is (div B)$d\tau$. Hence, by definition of divergence, eqn. 1.10 implies that, at all locations at all times,

$$\text{div}\, B = 0 \tag{1.11}$$