Grundlehren der mathematischen Wissenschaften 282

A Series of Comprehensive Studies in Mathematics

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Entire Functions of Several Complex Variables

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Springer-Verlag
Berlin Heidelberg NewYork Tokyo

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Mathematics Subject Classification (1980): 32A15

ISBN 3-540-15296-2
Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15296-2
Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging in Publication Data
Lelong, Pierre.
Entire functions of several complex variables.
(Grundlehren der mathematischen Wissenschaften; 282)
Bibliography: p.
Includes index.
1. Functions, Entire. 2. Functions of several complex variables. I. Gruman, Lawrence, 1942–.
II. Title. III. Series.
QA353.E5L44 1986 515.914 85-25028
ISBN 0-387-15296-2 (U.S.)

Printed in Germany

Typesetting, printing and bookbinding: Universitätsdruckerei H. Stürtz AG, Würzburg
2141/3140-543210

Introduction

I – Entire functions of several complex variables constitute an important and original chapter in complex analysis. The study is often motivated by certain applications to specific problems in other areas of mathematics: partial differential equations via the Fourier-Laplace transformation and convolution operators, analytic number theory and problems of transcendence, or approximation theory, just to name a few.

What is important for these applications is to find solutions which satisfy certain growth conditions. The specific problem defines inherently a growth scale, and one seeks a solution of the problem which satisfies certain growth conditions on this scale, and sometimes solutions of minimal asymptotic growth or optimal solutions in some sense.

For one complex variable the study of solutions with growth conditions forms the core of the classical theory of entire functions and, historically, the relationship between the number of zeros of an entire function $f(z)$ of one complex variable and the growth of $|f|$ (or equivalently $\log|f|$) was the first example of a systematic study of growth conditions in a general setting.

Problems with growth conditions on the solutions demand much more precise information than existence theorems. The correspondence between two scales of growth can be interpreted often as a correspondence between families of bounded sets in certain Fréchet spaces. However, for applications it is of utmost importance to develop precise and explicit representations of the solutions.

If we pass from $\mathbb{C}$ to $\mathbb{C}^n$, new problems such as problems of value distribution for holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^m$ arise. On the other hand, new techniques are often needed for classical problems to obtain solutions and representations of the solutions. Zeros of entire functions f are no longer isolated points; a measure of the zero set is obtained by the representation of the divisor X_f of f (and more generally of analytic subvarieties) by closed and positive currents, a class of generalized differential forms.

Paradoxally, it is the non-holomorphic objects, the "soft" objects (objets souples in French, see [C]) of complex analysis, principally plurisubharmonic functions and positive closed currents, which are adapted to problems with growth conditions, giving global representations in $\mathbb{C}^n$. Very often properties of the classical (i.e. holomorphic) objects will be derived from properties obtained for the soft objects. Plurisubharmonic functions were introduced in 1942 by K. Oka and P. Lelong. They occur in a natural way from the beginning of this book. Indicators of growth for a class of entire functions f are obtained as upper bounds for $\log|f|$; for $\log|f|$. To solve Cousin's Second Problem, i.e. to find (with growth conditions) an entire function f with given zeros X in $\mathbb{C}^n$, we solve first the general equation $i\partial\bar{\partial}V=\theta$ for a closed and positive current θ; if $\theta=[X]$, the current of integration on X, we then obtain f by $V=\log|f|$. Properties of plurisubharmonic functions appear again in a remarkable (and unexpected) result of H. Skoda (1972): there exists a representation for the analytic subvarieties Y in $\mathbb{C}^n$ of dimension $p(0\leq p\leq n-1)$ as the zero set $Y=F^{-1}(0)$ of an entire mapping $F=(f_1,\ldots,f_{n+1})$ such that $\|F\|$ is controlled by the growth of the area of Y. Plurisubharmonic functions obtained from potentials seem well adapted to the construction of global representations in $\mathbb{C}^n$; the method avoids the delicate study of ideals of holomorphic functions vanishing on Y and satisfying growth conditions.

The same methods using the soft object's properties of the current $(i\partial\bar{\partial}V)^p$ and the Monge-Ampère equation for plurisubharmonic functions V are employed for recent results obtained in value distribution theory of holomorphic mappings $\mathbb{C}^n\to\mathbb{C}^m$ or $X\to Y$, two analytic subvarieties in $\mathbb{C}^n$.

II - Before summarizing the content of this book, we would like to make some remarks.

a) We have not sought to give an exhaustive treatment of the subject (problems for $n>1$ are too numerous for a single book). We have tried to introduce the reader to the central problems of current research in this area, essentially that which had led to general methods or new technics. Applications appear only in Chapter 6 (to analytic number theory) and in Chapters 8 and 9 (to functional analysis).

b) On the other hand, we have tried to make the book self-contained. Some knowledge in the theory for one complex variable is required of the reader, as well as on integration, the calculus of differential forms and the theory of distributions. A list of books where the reader can find general results not developped here is given before the bibliography (such references are given by a capital roman letter).

The proofs of complementary results appear in three appendices: Appendix I for general properties of plurisubharmonic functions, Appendix II for the technic of proximate orders Appendix III for the $\bar{\partial}$ resolution for (0, 1) forms with L^2-estimates by Hörmander's method.

c) The importance of analytic representations, particulary for some applications, has made it necessary to give certain calculations *in extenso*. The authors are aware of the technical aspect of some developments given in the book. We recommand that the reader first read over the proof in order to assimilate the general idea before immersing himself in the details of the calculations.

d) The literature on the subject of entire functions is enormous. The bibliography, without pretending to be exhaustive, gives an overview of those areas of current interest. Each chapter has a short historical note which is an attempt to explain the origin of the given results.

III - Chapter 1 gives the basic definitions of the growth scales in $\mathbb{C}^n$, the notion of order and type, the indicator of growth and proximate orders. These classical notions extend trivially to plurisubharmonic functions and to entire functions in $\mathbb{C}^n$. In Chapter 2, we introduce the reader to the fundamental properties of positive differential forms and of positive and closed currents. Chapter 3 studies the solution with growth conditions of the equation $i\partial\bar{\partial}V=\theta$ for θ a positive closed current of type (1, 1) in $\mathbb{C}^n$, from which we deduce for $V=\log|f|$ the solution with growth conditions in $\mathbb{C}^n$ of Cousin's Second Problem and the representation of entire functions with a given zero set. The result for an entire function of finite order in $\mathbb{C}^n$ gives an extension of classical results of J. Hadamard and E. Lindelöf for $n=1$. Chapter 4 studies the class of entire functions f of regular growth. Certain results are given here for the first time. The importance of this study, which is based on the preceeding chapters, is in the numerous applications (Fourier transforms, differential systems) and the possibility of associating the regular growth of $\log|f|$ with the regular distribution of the zero set of f.

Chapter 5 studies the problems of entire maps $F: \mathbb{C}^n \to \mathbb{C}^m$. The first portion is devoted to the development of a representation of an analytic subvariety Y of $\mathbb{C}^n$ as the zero set of an entire map $F: \mathbb{C}^n \to \mathbb{C}^{n+1}$, that is $Y = F^{-1}(0)$, $F=(f_1, \ldots, f_{n+1})$, with control of the growth of the function $\|F\|$. The second part studies the growth of the fibers $F^{-1}(a) \cap B(0, r)$, where $B(0, r)=\{z: \|z\|<r\}$, when $r \to +\infty$. The third part studies the relationship between the growth of the area of an analytic set in $\mathbb{C}^n$ and its trace on linear subspaces of $\mathbb{C}^n$. The cases of slow growth and algebraic growth are also studied.

Chapter 6 gives an example of an application of the methods of the preceeding chapters to a problem in number theory. We show that the set of points of $\mathbb{C}^n$ where certain families of meromorphic functions of finite order take on algebraic values is contained in an algebraic subvariety of $\mathbb{C}^n$ whose degree can be bounded: this famous result of E. Bombieri (1970) gave a very deep and unexpected application of the theory of closed positive currents t and of the number $v_t(x)$ (a kind of multiplicity for x on the support of t) to number theory, via a classical method of Siegel and L^2-estimates for the $\bar{\partial}$ operator. The same idea was also fundamental some time later in Siu's Theorem about the structure of closed positive currents.

Chapter 7 establishes the theory of the indicator of growth theorem for entire functions of finite order in $\mathbb{C}^n$: every plurisubharmonic function positively homogeneous of order ρ is the (regularized) indicator of growth function of an entire function of order ρ.

Chapters 8 and 9 concern applications of entire functions to classes of linear operators. Indeed the space $\hat{\mathscr{D}}(\Omega)$ of the Fourier transforms of the distributions defined in a bounded domain Ω of $\mathbb{C}^n$ is a subspace of the space $\mathscr{H}(\mathbb{C}^n)$ of the entire functions in $\mathbb{C}^n$, and many problems characterize classes of distributions in Ω by growth properties of the image in $\mathscr{H}(\mathbb{C}^n)$. This method leads to analytic functionals. The analytic functionals are the elements of the dual space of the space $\mathscr{H}(\Omega)$ of holomorphic functions in Ω, equipped with the topology of uniform convergence on compact subsets of Ω. Chapter 8 gives a study of the Fourier-Borel transform and of the Laplace transform in order to obtain properties for analytic functionals and their supports.

Chapter 9 gives a general treatment of convolution operators in linear spaces of entire functions. New results in particular for the functions of order $\rho<1$ are given as consequences of the techniques developped in preceeding sections of the book.

We use the following system of notations for references: a statement (theorem, lemma, proposition, definition etc.) is given two numbers, the first indicating the chapter in which it is found and the second indicating its position in that chapter. Thus, Theorem 8.23 refers to the 23-rd statement in Chapter 8. Figures within parentheses refer to equations in the text, for instance (4, 18) refers to the eighteenth equation in Chapter 4. Roman numerals I, II, III, refer to the three appendices which are at the end of the book.

The authors thank K. Diederich, R. Gay, H. Skoda and M. Waldschmidt who read different parts of the manuscript and whose remarks were fruitful for many improvements. They are indebted to Mireille Geurts for typing the manuscript, which was no small nor easy task. And we should like to thank the editors of Springer-Verlag for their excellent and rapid preparation of the book.

Paris and Marseille,
January 1986

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Chapter 1. Measures of Growth

§ 1. Preliminaries

We will let $\mathbb{C}$ represent the field of complex numbers and $\mathbb{R}$ the subfield of real numbers. Let $z=(z_1, \ldots, z_n)$ be an element of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, the underlying space of real coordinates. The transformations from the complex to the real coordinates are given by $z_k = x_k + iy_k$, $\bar{z}_k = x_k - iy_k$ and $x_k = \frac{z_k + \bar{z}_k}{2}$, $y_k = \frac{z_k - \bar{z}_k}{2i}$. Unless specified to the contrary, we equip $\mathbb{C}^n$ with the Euclidian metric of $\mathbb{R}^{2n}$:

$$(1,1) \qquad ds^2 = \sum_{k=1}^{n} (dx_k^2 + dy_k^2) = \sum_{k=1}^{n} dz_k \cdot d\bar{z}_k$$

and we choose for $\mathbb{C}^n$ the volume form

$$\tau = \bigwedge_{k=1}^{n} (dx_k \wedge dy_k) = (i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n.$$

By a domain Ω, we shall always mean an open connected set. We let $d_\Omega(z)$, the distance to the boundary, be defined for $z \in \Omega$ by $d_\Omega(z) = \inf_{z' \notin \Omega} \|z - z'\|$ (where $\| \ \|$ represents the Euclidean norm) and set $d_\Omega(z) = +\infty$ if $\Omega = \mathbb{C}^n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-indice of non-negative integers. We then define $|\alpha|$ by $|\alpha| = \sum_{i=1}^{n} \alpha_i$, the differential operator D^α by $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}$, and z^α by $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$.

We let $\mathscr{C}^k(\Omega)$ be the set of functions defined on Ω all of whose derivatives up to order $|\alpha| \leq k$ are continuous and $\mathscr{C}^\infty(\Omega)$ the set of functions whose derivatives of all orders are continuous. By $\mathscr{C}_0^k(\Omega)$ (resp. $\mathscr{C}_0^\infty(\Omega)$) we will mean the subset of $\mathscr{C}_0^k(\Omega)$ (resp. $\mathscr{C}_0^\infty(\Omega)$) composed of those functions whose support in Ω is compact. We let ∂ and $\bar{\partial}$ be the exterior differential operators defined by

$$\partial = \sum_{k=1}^{n} \frac{\partial}{\partial z_k} dz_k, \qquad \bar{\partial} = \sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k} d\bar{z}_k$$

and set

$$d = \partial + \bar{\partial} = \sum_{k=1}^{n} \left(\frac{\partial}{\partial x_k} dx_k + \frac{\partial}{\partial y_k} dy_k \right).$$

A function $f\colon \Omega\subset\mathbb{C}^n\to\mathbb{C}$ is said to be holomorphic in Ω if $f\in\mathscr{C}^1(\Omega)$ and $\bar\partial f=0$. In particular, this means that $\frac{\partial f}{\partial \bar z_k}=0$ for $1\leqq k\leqq n$. The domain $\Delta(z',r)=[z\colon |z'_k-z_k|<r_k,\ r_k>0,\ 1\leqq k\leqq n]$ is called the *polydisc of center* z', of radii r_k. For $\Delta(z',r)\Subset\Omega$, and f holomorphic in Ω, the iteration of the Cauchy Integral Formula for one complex variable gives for $z\in\Delta(z',r)$ the *integral representation:*

$$(1,2)\qquad f(z)=(2\pi)^{-n}\int_0^{2\pi}\dots\int_0^{2\pi}\frac{f(z'_1+r_1e^{i\theta},\dots,z'_n+r_ne^{i\theta})}{\prod_{k=1}^{n}(z'_k+r_ke^{i\theta_k}-z_k)}\,d\theta_1\dots d\theta_n.$$

As for $n=1$, we deduce from (1, 2) a Taylor series expansion:

$$f(z)=\sum_{(\alpha)}C_\alpha(z-z')^\alpha,\qquad \alpha=(\alpha_1,\dots,\alpha_n),$$

which converges uniformly for $|z'_k-z_k|\leqq r'_k<r_k$. Then we obtain a Taylor series expansion on each compact polydisc of Ω. We designate by $\mathscr{H}(\Omega)$ the family of functions holomorphic in Ω. By an *entire function*, we shall mean an element of $\mathscr{H}(\mathbb{C}^n)$. Thus, an entire function $f(z)$ has a Taylor series expansion $f(z'+z)=\sum_m\sum_{|\alpha|=m}P_\alpha(z')z^\alpha$ which, for every point z', converges uniformly in z on compact subsets of $\mathbb{C}^n$. We say that $\sum_{|\alpha|=m}P_\alpha(z')z^\alpha$ is the homogeneous polynomial of degree m in the Taylor series expansion of $f(z)$ at the point z'.

§ 2. Subharmonic and Plurisubharmonic Functions

In our study of entire functions f of several complex variables, we shall be interested in the asymptotic growth of $|f|$, $f\in\mathscr{H}(\mathbb{C}^n)$, or equivalently by the asymptotic growth of $\log|f|$. Suppose for instance that $\varphi(t)$ is an increasing function of t for $t\geqq 0$ such that $\limsup_{t\to\infty}\frac{\varphi(tu)}{\varphi(t)}<\infty$, for $u\geqq 0$ and $\lim_{t\to\infty}\varphi(t)=\infty$, and consider in $\mathscr{H}(\mathbb{C}^n)$ the subclass M_φ defined by the condition

$$\log|f(z)|\leqq\varphi(\|z\|)+C(f).$$

Then the function

$$\chi_f(z)=\limsup_{t\to\infty}\frac{\log|f(tz)|}{\varphi(t)}$$

measures an asymptotic growth with respect to the weight factor $\varphi(t)$ on the real lines through the origin. Thus we are led to consider expressions of the form $\limsup_{i\in I}C_i\log|f_i|$, $f_i\in\mathscr{H}(\mathbb{C}^n)$, $C_i\in\mathbb{R}^+$. This leads us to study filtered families included in a larger class of functions, the plurisubharmonic functions introduced by K. Oka and P. Lelong. This family is closed under the operation of taking the smallest upper semi-continuous majorant of a filtered family uniformly bounded above on compact subsets (in fact, one can show that the functions $C\log|f|$, $f\in\mathscr{H}(\mathbb{C}^n)$, $C\in\mathbb{R}^+$, generate locally the plurisubharmonic functions under this operation, but we shall not need this property). The plurisubharmonic functions play the same role for $n>1$ complex variables that the subharmonic function play in complex analysis of one complex variable. Moreover, in $\mathbb{C}^n$, for $n\geqq 2$, the growth of entire functions has properties which can be compared to the classical properties (pseudo-convexity, sometimes R^n-convexity) of domains of holomorphy. The use of plurisubharmonic functions and the systematic exploitation of their properties will lead to most of the results in this book.

We begin by recalling important definitions. The proofs of the properties that we shall need can be found in Appendix I (referred to by App. I).

Definition 1.1. Let $\Omega\subset\mathbb{R}^m$ be a domain. A real valued function $\varphi(-\infty\leqq\varphi(x)<+\infty)$ is said to be subharmonic in Ω if

a) φ is upper semi-continuous and $\varphi(x)\not\equiv-\infty$ in Ω,

b) $\varphi(x)\leqq\lambda(x,r,\varphi)\equiv\omega_m^{-1}\int\varphi(x+r\alpha)d\omega_m(\alpha)$ for $r<d_\Omega(x)$ where ω_m is the Lebesgue measure of the unit sphere S^{m-1} in $\mathbb{R}^m$ and λ is the mean value of φ on S^{m-1} relative to the Haar measure $d\omega_m(\alpha)$.

Definition 1.2. Let $\Omega\subset\mathbb{C}^n$ be a domain. A real valued function $\varphi(-\infty\leqq\varphi(z)<+\infty)$ is said to be *plurisubharmonic* in Ω if it has property (a) above and in addition b$_2$) $\varphi(z)\leqq l(z,w,r,\varphi)=\dfrac{1}{2\pi}\int_0^{2\pi}\varphi(z+wre^{i\theta})d\theta$ for all w, r such that $z+uw\subset\Omega$ for $u\in\mathbb{C}$, $|u|\leqq r$.

In the sequel we denote by $D(z,w,r)$ the compact disc

$$\{z'\in\mathbb{C}^n\colon z'=z+uw,\ u\in\mathbb{C},\ |u|\leqq r\}.$$

We shall let $S(\Omega)$ designate the set of subharmonic functions defined on a domain $\Omega\subset\mathbb{R}^m$ and by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic functions defined on a domain $\Omega\subset\mathbb{C}^n$. We recall some classical properties of the sets $S(\Omega)$ and $\mathrm{PSH}(\Omega)$ (we refer the reader to Appendix I for the proofs):

i) if τ_m is the volume of the unit ball in $\mathbb{R}^m$ and $\varphi\in S(\Omega)$, then $\varphi(x)\leqq\tau_m^{-1}r^{-m}\int_{\|x'\|\leqq r}\varphi(x+x')d\tau(x')=A(x,r,\varphi)$ for $r<d_\Omega(x)$ (cf. Remark after Definition I.1);

ii) $S(\Omega)\subset L^1_{\mathrm{loc}}(\Omega)$, the family of locally Lebesgue integrable functions, and $\mathrm{PSH}(\Omega)\subset S(\Omega)$ for $\Omega\subset\mathbb{C}^n$ (Proposition I.9).

iii) the set $\{x\in\Omega\colon\varphi(x)=-\infty,\ \varphi\in S(\Omega)\}$ is of Lebesgue measure zero in Ω (Corollary I.12);

iv) for $\varphi\in S(\Omega)$ and $x\in\Omega$ either $\varphi(x)<\sup_\Omega\varphi(x)$ or φ is a constant (Proposition I.13);

v) if $\varphi \in \mathrm{PSH}(\Omega)$ and $D(z,w,r)\subset\Omega$, then the two functions $r\to l(z,w,r,\varphi)$ and $r\to \sup_{z\in D(z,w,r)} \varphi(z)$ are increasing convex functions of $\log r$, and as a consequence $\lambda(z,r,\varphi)$ and $M(z,r,\varphi)=\sup_{\|z'-z\|\leq r}\varphi(z')$ are increasing convex functions of $\log r$ (Proposition I.17);

vi) if $F:\Omega\to\Omega'$ is a holomorphic homeomorphism (analytic isomorphism) of Ω onto Ω', then the map $T:\varphi\in\mathrm{PSH}(\Omega)\to\varphi\circ F^{-1}\in\mathrm{PSH}(\Omega')$ is a bijection (i.e. the class of plurisubharmonic functions is invariant with respect to holomorphic homeomorphisms and is thus an object of the analytic structure; this is false for the larger class $S(\Omega)$).

The preceeding remarks shall play a crucial role, since for $f\in\mathscr{H}(\mathbb{C}^n)$, the functions $\log|f|$ form a subset $V(\mathbb{C}^n)$ of $\mathrm{PSH}(\mathbb{C}^n)$. But the class $\mathrm{PSH}(\mathbb{C}^n)$ also contains certain functions which do not belong to $V(\mathbb{C}^n)$ (such as convex functions in the space of the variables $(\log r_1,\ldots,\log r_n)$) (cf. App. I). What is more, it is fruitful to introduce certain measure theoretic concepts in the class $\mathrm{PSH}(\mathbb{C}^n)$.

On $\mathrm{PSH}(\Omega)$ and $S(\Omega)$, we consider the topology $L^1_{\mathrm{loc}}(\Omega)$ defined by the seminorms $N_K(\varphi)=\int_K|\varphi(z)|\,d\tau(z)$ where $d\tau$ is the Lebesgue measure and K is compact in Ω. Actually, it is sufficient to consider the semi-norms $N_i(\varphi)=\int_{B_i}|\varphi(z)|\,d\tau(z)$ where B_i runs over a countable family of compact balls which cover Ω (in $L^1_{\mathrm{loc}}(\Omega)$, we do not distinguish between two functions which are equal almost everywhere). We note that with this topology, $L^1_{\mathrm{loc}}(\Omega)$ is a Fréchet space; $S(\Omega)$ and $\mathrm{PSH}(\Omega)$ are convex cones in $L^1_{\mathrm{loc}}(\Omega)$, closed for this topology (see App. I).

Theorem 1.3. *A subset $M\subset S(\Omega)$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$ if and only if the elements of M have a common upper bound on every compact subset of Ω and if M does not contain a sequence which converges uniformly to $-\infty$ on each compact set $K\subset\Omega$.*

Proof. If M is bounded in $L^1_{\mathrm{loc}}(\Omega)$, then it does not contain a sequence φ_k which tends uniformly to $-\infty$ on any compact ball B, since the integrals $\int_B|\varphi_k|\,d\tau$ are uniformly bounded. Let K be a compact subset of Ω and define K' by $K'=\bigcup_{x\in K}\bar{B}(x,\frac{1}{2}\delta_K)$ where $\delta_K=\inf_{x\in K}[d_\Omega(x),1]$.

Then K' is compact in Ω and for $x\in K$ and $\varphi\in S(\Omega)$ we have

$$\varphi(x)\leqq A(x,\tfrac{1}{2}\delta_K,\varphi)\leqq\tau_m^{-1}2^m\delta_K^{-m}\int_{K'}|\varphi(x)|\,d\tau(x).$$

Thus, if M is bounded, the elements of M have a common upper bound on every compact subset of Ω.

Conversely, let $M\subset S(\Omega)$ and suppose that M is uniformly bounded on every compact subset of Ω. If there exists a semi-norm N_i such that $\{N_i(\varphi),\ \varphi\in M\}$, is not bounded, we can find a sequence $\varphi_k\in M$ such that $N_i(\varphi_k)\to\infty$, and since the φ_k are uniformly bounded above on B_i, $\lim_{k\to\infty}\int_{B_i}\varphi_k d\tau = -\infty$. Let $\delta>0$ be the distance of $B_i=B(x_i,r_i)$ to $\complement\Omega$, $\delta=\inf_{x'\in B_i} d_\Omega(x')$, and let α be such that $0<2\alpha<\delta$. For $x\in B(x_i,\alpha)$, we have

$$B_i\subset B(x_i,r_i+\alpha)\Subset B_i'=B(x_i,r_i+2\alpha)\Subset\Omega.$$

If σ is an upper bound for φ_k in B_i', then for $x\in B(x_i,\alpha)$, we obtain by the Mean Value Property for subharmonic functions:

$$\begin{aligned}\varphi_k(x)-\sigma &\leqq A(x,r_i+\alpha,\varphi_k-\sigma)\leqq[\tau_m(r_i+\alpha)^m]^{-1}\int_{B_i}[\varphi_k(x)-\sigma]\,d\tau_m\\ &\leqq\tau_m^{-1}(r_i+\alpha)^{-m}\Big[C\sigma+\int_{B_i}\varphi_k(x')\,d\tau_m(x')\Big]\end{aligned}$$

which proves the uniform convergence of the sequence φ_k to $-\infty$ on $B(x_i,\alpha)$.

Let $\hat\Omega$ be the largest open subset of Ω such that $\{\varphi_k\}$ converges to $-\infty$ uniformly on every compact subset of $\hat\Omega$. Since $B(x_i,\alpha)\subset\hat\Omega$, we know that $\hat\Omega$ is not empty. Moreover, if x' is a limit point of $\hat\Omega$ in Ω, there exists a ball $B(x',\rho)\Subset\Omega$ such that $B(x',\rho)\cap\hat\Omega$ contains a set $K\Subset\Omega$ of positive Lebesgue measure. Then $\lim_{k\to\infty}\int_{B(x',\rho)}\varphi_k d\tau=-\infty$, and by the above reasoning, $\varphi_k(x)\to-\infty$ uniformly on $B(x',\alpha)$ for $\alpha>0$ such that $\rho+\alpha<d_\Omega(x')$. Thus $x'\in\hat\Omega$ so $\hat\Omega$ is closed. Since $\hat\Omega$ is open, closed, and is a domain, $\hat\Omega=\Omega$. □

§3. Norms on $\mathbb{C}^n$ and Order of Growth

Let $p(z)$ be a real valued function on $\mathbb{C}^n$. We say that $p(z)$ is *subadditive* if $p(z+z')\leqq p(z)+p(z')$; we say that $p(z)$ is *positively homogeneous* of order ρ if $p(tz)=t^\rho p(z)$ for $t\geqq 0$; we say that $p(z)$ is *complex homogeneous* of order ρ if $p(uz)=|u|^\rho p(z)$, $u\in\mathbb{C}$. If $p(z)$ is subadditive and $p(tz)=|t|\,p(z)$ for $t\in\mathbb{R}$ (resp. $p(\lambda z)=|\lambda|\,p(z)$ for $\lambda\in\mathbb{C}$), we say that $p(z)$ is a *real (resp. complex) semi-norm*. If, in addition, $p(z)=0$ if and only if $z=0$, then $p(z)$ is a *real* (resp. *complex*) norm.

If $p(z)$ is a norm, we define the p-ball of center z and radius r by $B_p(z,r)=\{z':p(z-z')<r\}$, and if the norm is not specified, it will be assumed to be the Euclidean norm $\|z\|$. We recall that if p and q are two norms on $\mathbb{C}^n$, then each determines the unique separated vector space topology on $\mathbb{C}^n$, and there exist positive finite constants C_1 and C_2 such that

(1,3) $$0<C_1\leqq\frac{p(z)}{q(z)}\leqq C_2.$$

Given a function $a(z): \mathbb{C}^n \to \mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$, we consider

$$M_{a,p}(r) = \sup_{p(z) \leqq r} a(z). \tag{1,4}$$

It then follows from (1,3) that there exist constants C and C', $0 < C < C' < \infty$, depending only on $p(z)$ and $q(z)$, such that for every real valued function $a(z)$

$$M_{a,q}(Cr) \leqq M_{a,p}(r) \leqq M_{a,q}(C'r). \tag{1,5}$$

The functions we shall consider will often be plurisubharmonic, and in this case we have:

Proposition 1.4. *If $\varphi(z)$ is plurisubharmonic in $\mathbb{C}^n$, then*

a) *$m_\varphi(z, z', r) = \sup_{|u| \leqq r} \varphi(z + uz')$ is identically $-\infty$ or an increasing convex function of* $\log r$;

b) *if $p(z)$ is a complex norm, then $M_{\varphi,p}(r)$ is an increasing convex function of* $\log r$.

Proof. For a): $\varphi(z + uz') = -\infty$ for all $u \in \mathbb{C}$ or $\varphi(z + uz')$ is a subharmonic function of the variable $u = \alpha + i\beta$ in $\mathbb{C} = \mathbb{R}^2$ (cf. Remark 2 after Definition I.2).

For b): Consider $M_{\alpha,p}(r) = \sup_{z \in p^{-1}(1)} [\sup_{|u| \leqq r} \varphi(uz)]$ and remark that $\sup_{|u| \leqq r} \varphi(uz)$ is an increasing convex function of $\log r$ or identically $-\infty$, but is not identically $-\infty$ for all z. □

§4. Minimal Growth: Liouville's Theorem and Generalizations

The existence of a minimal growth for a non-constant function $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ is just a consequence of the convexity properties of Proposition 1.4 and formula (1,5).

Theorem 1.5. i) *Let $p(z)$ be a norm and $\varphi(z)$ a plurisubharmonic function in $\mathbb{C}^n$. Then $C = \lim_{r \to \infty} \frac{M_{\varphi,p}(r)}{\log r}$ and $C(z, z') = \lim_{r \to \infty} \frac{M_\varphi(z, z', r)}{\log r}$ exist, either finite or infinite, with the following properties:*

a) *$C \geqq 0$; moreover $C(z, z') \geqq 0$ with the possible exception $C(z, z') = -\infty$ in which case $\varphi(z + uz') \equiv -\infty$ for $u \in \mathbb{C}$.*

b) *$C(z, uz') \equiv C(z, z')$ for every $u \in \mathbb{C}$, $u \neq 0$.*

ii) *if* $p(z)$ *is a norm on* $\mathbb{C}^n$ *and* $\varphi(z+uz')\not\equiv -\infty$, $u\in\mathbb{C}$, *then* $\frac{\partial}{\partial \log r} M_{\varphi,p}(r)$ *and* $\frac{\partial}{\partial \log r} m_\varphi(z,z',r)$ *exist except perhaps for a countable set of r and*

$$\lim_{r\to\infty} \frac{\partial}{\partial \log r} M_{\varphi,p}(r) = \lim_{r\to\infty} \frac{M_{\varphi,p}(r)}{\log r};$$

$$\lim_{r\to\infty} \frac{\partial m_\varphi(z,z',r)}{\partial \log r} = \lim_{r\to\infty} \frac{m_\varphi(z,z',r)}{\log r}.$$

Proof. There exists $r_0>0$ such that $M_{\varphi,p}(r_0)>-\infty$, and since it is an increasing convex function of $\log r$, $M_{\varphi,p}(r)>-\infty$ for $r\geqq r_0$, which proves that $C\geqq 0$. If $\varphi(z+uz')\equiv -\infty$, then for $r>1$, $(\log r)^{-1} m_\varphi(z,z',r)=-\infty$, hence $C(z,z')=-\infty$. Otherwise, $\varphi(z+uz')$ is an $\mathbb{R}^2$-subharmonic function of u and hence by the above reasoning. $C(z,z')\geqq 0$. From the definition, by an obvious calculation we obtain $C(z,z')=C(z,uz')$ for $u\neq 0$. Part (ii) follows directly from Proposition 1.4. □

Theorem 1.6. *Suppose* $f\in\mathscr{H}(\mathbb{C}^n)$ *and set* $\varphi(z)=\log|f(z)|$. *Let*

$$C(z')=\liminf_{r\to\infty}\frac{m_\varphi(0,z',r)}{\log r}=\lim_{r\to\infty}\frac{m_\varphi(0,z',r)}{\log r}.$$

Then

i) $\eta_\rho=\{z';\,C(z')\leqq\rho\}$ *is a cone and if* η_ρ *is not contained in an algebraic hypersurface defined as the zero set of a homogeneous polynomial of degree* $\rho'\leqq\rho$, *then f is a polynomial of degree at most* ρ;

ii) *if f is a polynomial of degree m, then*

$$C'_m(z')=\liminf_{r\to\infty}\,[m_\varphi(0,z',r)-m\log r]=m_\psi(0,z',1)$$

where $\psi=\log|P_m|$ *and* P_m *is the homogeneous polynomial of maximal degree m in f. Furthermore for* $z'\in\mathbb{C}^n-\{0\}$ *we have*

$$\{z';\,C'_m(z')=-\infty\}=\{z';\,P_m(z')=0\}=\{z';\,C(z')\neq m\}.$$

Proof. Let $f(z')=\sum_{k=0}^{\infty}P_k(z')$ be the Taylor series expansion of $f(z')$ in terms of homogeneous polynomials. Then for $u\in\mathbb{C}$, $f(uz')=\sum_{k=0}^{\infty}P_k(z')u^k$. It follows from the Cauchy Integral Formula that $P_k(z')=1/2\pi\int_0^{2\pi} f(re^{i\theta}z')r^{-k}e^{-ik\theta}\,d\theta$, and hence $\log|P_k(z')|\leqq m_\varphi(0,z',r)-k\log r$. Thus, if there exists a sequence $r_\mu(z')\to\infty$ such that $[m_\varphi(0,z',r_\mu(z'))-k\log r_\mu(z')]\to-\infty$, we have $P_k(z')=0$. So if η_ρ is not contained in the zeros of a homogeneous polynomial of degree $\rho'\leqq\rho$, then $P_k(z')\equiv 0$ for $k>\rho$. If $f(z)$ is a polynomial of degree m, then $|f(rz')|\leqq|P_m(z')r^m|+O(r^{m-1})$, from which the second part follows. □