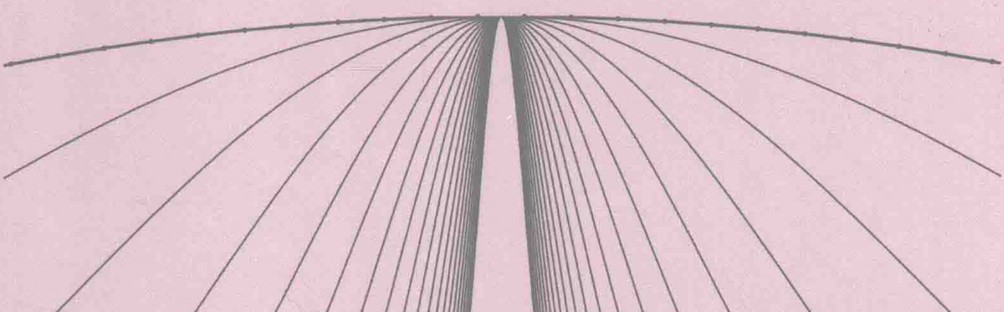
THE FRACTIONAL LAPLACIAN



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THE FRACTIONAL LAPLACIAN

Preface

The ordinary Laplacian is defined as the ordinary second derivative of a function of one variable or the sum of the ordinary second partial derivatives of a function of a higher number of variables in a physical or abstract Caresian space. Physically, the ordinary Laplacian describes an ordinary diffusion process in an isotropic medium mediated by non-idle random walkers who step into neighboring or nearby sites of an idealized grid, but are unable to perform long jumps.

In the physical sciences, the ordinary Laplacian appears as a contribution to a conservation law or evolution equation due to a diffusive species flux according to Fick's law, a conductive thermal flux according to Fourier's law, or a viscous stress according to the Newtonian constitutive equation. An implied assumption is that the rate of transport of a field of interest at a certain location is determined by an appropriate field variable at that location, independent of the global structure of the transported field.

The fractional Laplacian, also called the Riesz fractional derivative, describes an unusual diffusion process due to random displacements executed by jumpers that are able to walk to neighboring or nearby sites, and also perform excursions to remote sites by way of Lévy flights. Literal or conceptual flights have been observed or alleged to occur in a variety of applications, including turbulent fluid motion and material transport in fractured media. In the context of mechanics, the fractional Laplacian describes the motion of a chain or array of particles that are connected by elastic springs not only to their nearest neighbors, but also to all other particles. The spring constant diminishes with the particle separation, while the particle array may describe an ordinary or fractal configuration.

A key physical concept underlying the notion of the fractional Laplacian is the fractional diffusive flux, arising as a generalization of the ordinary diffusive flux expressed by Fick's law, the ordinary conductive flux expressed by Fourier's law, or the expression for the viscous stress according to the Newtonian constitutive equation. The generalized flux associated with the fractional Laplacian provides us with expressions for the rate of transport at a certain location as an integral of an appropriate field variable over an appropriate domain of influence. The fractional diffusive flux at a certain location is affected by the state of the field in the entire space.

The extraordinary effect of the fractional flux can be demonstrated by considering species diffusion or heat conduction in two isolated patches that are separated by an insulating material. Assume that the first patch is devoid of a diffusing species, or else isothermal, whereas the second patch hosts a diffusive

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species, or else supports a temperature field. Under the influence of a fractional flux, the first patch develops a concentration or temperature field due to the second patch in a process that may appear as an optical illusion or an instance of the paranormal. The physical reason is that material and energy can be transported over long distances by physical or conceptual splattering.

In the most general abstract context, the fractional Laplacian describes the contribution to a conservation law of a non-local process that is affected not only by the local conditions, but also by the global state of a field of interest at a given time. Non-local dependencies are familiar to those who study non-Newtonian mechanics. Applications can be envisioned in a broad range of disciplines in mainstream science and engineering, image processing, but also in sociology, entomology, health care management, and finance.

The notion of the fractional Laplacian provides us with an interesting tool for mathematical modeling when traditional approaches appear to fail. The subtlety of the underlying mathematical concepts has motivated a substantial body of literature in applied mathematics and selected physical sciences. Despite a long history and considerable progress made in recent years, the general subject is still emerging and a number of conceptual and computational issues require further elaboration.

My goal in this book is to offer a concise introduction to the fractional Laplacian at a level that is accessible to mainstream scientists and engineers with a rudimentary background in ordinary differential and integral calculus. Emphasis is placed on fundamental ideas and practical numerical computation. Original material is included throughout the book and novel numerical methods are developed.

There are two intentional peculiarities in the presentation. First, the fractional Laplacian in three dimensions is discussed in Chapter 5, followed by the fractional Laplacian in two dimensions in Chapter 6, and then followed by the fractional Laplacian in arbitrary dimensions in Appendix D. This ordering is due to certain unusual properties of Laplace's equation in two dimensions coupled with the author's belief that the most general case should not necessarily be treated first. The second peculiarity relates to the occasional near-repetition of discussion and equations in one, two, or three dimensions. Although consolidation would have abbreviated the discourse, it would have compromised the reader's ability to study the material in a non-sequential fashion.

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Notation

$\dot{f}$	ordinary first derivative of a function, $f(x)$
$\ddot{f}$	ordinary second derivative of a function, $f(x)$
$f^{(n)}$	ordinary nth derivative of a function, $f(x)$
f'	fractional first derivative of a function, $f(x)$
f''	fractional Laplacian of a function, $f(x)$
f'''	fractional third derivative of a function, $f(x)$
f'''	fractional fourth derivative of a function, $f(x)$
$\widehat{f}$	Fourier transform of a function, $f(x)$ or $f(\mathbf{x})$
$\nabla^\alpha f$	fractional Laplacian of a function, $f(\mathbf{x})$
$\nabla^2 f$	ordinary Laplacian of a function, $f(\mathbf{x})$
$\boldsymbol{\nabla} f$	ordinary gradient of a function, $f(\mathbf{x})$
$\boldsymbol{\nabla}^{\alpha-1} f$	fractional gradient of a function, $f(\mathbf{x})$
$\boldsymbol{\nabla} \cdot \mathbf{f}$	ordinary divergence of a function, $f(\mathbf{x})$
$\nabla^\alpha f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}^{\alpha-1} f$	fractional Laplacian of a function, $f(\mathbf{x})$
$\mathrm{F}(\alpha, \beta; \gamma; z)$	Gauss hypergeometric function
ζ_s	Riemann zeta function
$\mathrm{B}(\xi)$	Beta function
$\Gamma(\xi)$	Gamma function
ϕ_α	Laplacian potential, $\nabla^\alpha f = \nabla^2 \phi_\alpha$
$\Phi(a, b; z)$	degenerate hypergeometric function
$c_{d,\alpha}$	coefficient in front of the principal-value integral defining the fractional Laplacian in d dimensions
$\epsilon_{d,\alpha}^{(1)}$	coefficient in front of the integral defining the fractional derivative or gradient in d dimensions
$\epsilon_{d,\alpha}^{(0)}$	coefficient in front of the integral defining the Laplacian potential of the fractional Laplacian in d dimensions
$\epsilon_{d,\alpha}^{(3)}$	coefficient in front of the integral defining the fractional third derivative in d dimensions
$\epsilon_{d,\alpha}^{(4)}$	coefficient in front of the integral defining the fractional fourth derivative in d dimensions
$\epsilon_{d,\alpha}^{(m)}$	coefficient in front of the integral defining the fractional mth derivative in d dimensions
ℓ	characteristic length
τ	characteristic time
κ_α	fractional diffusivity

Contents

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1 The fractional Laplacian in one dimension

In the first chapter, we provide physical motivation for the fractional Laplacian of a function of one variable in the context of random walks underlying diffusion, provide a rigorous definition for the fractional Laplacian in terms of the Fourier transform or a principal-value integral, and discuss the Green's function of the fractional Laplace equation and of the unsteady fractional diffusion equation. Numerical methods for solving differential equations involving the fractional Laplacian are developed in Chapter 2, and further concepts in one dimension are discussed in Chapter 3.

1.1 Random walkers with constant steps

Consider a column of N_p point particles sitting on the x axis at the position $x = \frac{1}{2}\Delta x$, and another column of N_p point particles sitting on the x axis at the mirror image position, $x = -\frac{1}{2}\Delta x$, where Δx is a specified interval, as shown in Figure 1.1.1(a). The total number of particles is $2N_p$.

At the origin of computational time, each particle starts making random steps to the *right* with probability q or to the *left* with probability $1 - q$, where q is a free parameter in the range $0 \le q \le 1$. After one step has been made, each particle has been displaced to the left or to the right by a fixed distance, Δx.

After n steps have been made, the particles have spread out from the two initial columns to occupy discrete positions along the x axis located at the half-integer nodes

$$x_i = \left(i - \tfrac{1}{2}\right)\Delta x \tag{1.1.1}$$

for $i = 0, \pm 1, \pm 2, \ldots$, where the ith node hosts $m_i(n)$ particles, as shown in Figure 1.1.1(b). The initial condition specifies that $m_i(0) = 0$ for any i, except that

$$m_0(0) = N_p, \qquad m_1(0) = N_p \tag{1.1.2}$$

describing the two columns. Since the particles move by a fixed distance Δx at every step, $m_i(n) \neq 0$ only for $-n \le i \le n+1$. Particle number conservation

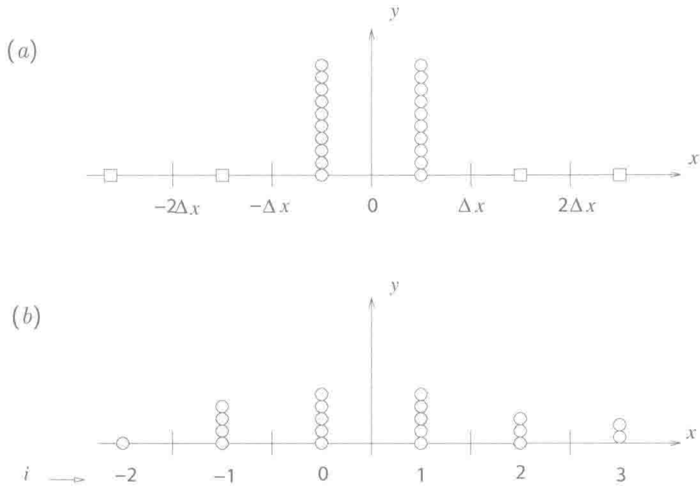

FIGURE 1.1.1 (a) Initial and (b) subsequent distribution of random walkers moving with constant step along the x axis at notched positions.

requires that

$$\sum_{i=-n}^{n+1} m_i(n) = 2N_p \tag{1.1.3}$$

after any number of steps, n.

We may introduce an arbitrary time step, Δt, and regard

$$t_n = n\Delta t \tag{1.1.4}$$

as time elapsed, providing us with a time series.

As a technicality, we note that, if all particles were placed in a single file at the origin at the initial time, $x = 0$, they would occupy odd- and even-numbered positions at later times, which is somewhat counterintuitive but not essentially alarming.

1.1.1 Particle number density distribution

To study the collective particle motion, we quantify the population dynamics in terms of a discrete number-density distribution defined as

$$p_i(n) \equiv \frac{1}{2N_p}\, m_i(n) \tag{1.1.5}$$

for $i = -n, \ldots, n+1$, which may also be regarded as a discrete probability density function (dpdf). By definition, and because of particle conservation,

$$\sum_{i=-n}^{n+1} p_i(n) = 1 \tag{1.1.6}$$

independent of the number of steps made, n. The initial condition specifies that

$$p_0(0) = \tfrac{1}{2}, \qquad p_1(0) = \tfrac{1}{2}, \tag{1.1.7}$$

describing the two columns in Figure 1.1.1(a).

Expected position and variance

Two variables of interest are: (a) the expected particle position after n steps given by

$$\overline{x}(n) \equiv \sum_{i=-n}^{n+1} x_i\, p_i(n), \tag{1.1.8}$$

and (b) the associated variance defined as the square of the standard deviation, $s(n)$, according to the equation

$$s^2(n) \equiv \sum_{i=-n}^{n+1} \big(x_i - \overline{x}(n)\big)^2 p_i(n). \tag{1.1.9}$$

Expanding the square, we obtain

$$s^2(n) = \sum_{i=-n}^{n+1} x_i^2\, p_i(n) - 2\,\overline{x}(n) \sum_{i=-n}^{n+1} x_i\, p_i(n) + \overline{x}^2(n) \sum_{i=-n}^{n+1} p_i(n). \tag{1.1.10}$$

Consolidating the second and third terms on the right-hand side, we obtain

$$s^2(n) = -\overline{x}^2(n) + \sum_{i=-n}^{n+1} x_i^2\, p_i(n). \tag{1.1.11}$$

The sum on the right-hand side can be computed even before the expected particle position is available.

Physical intuition suggests that the expected particle position evolves linearly in time at a rate given by

$$\frac{\mathrm{d}\overline{x}}{\mathrm{d}t} = \frac{2q-1}{\Delta t}. \tag{1.1.12}$$

For example, when $q = \frac{1}{2}$, the expected position remains constant, as each particle has the same probability of moving to the right or left at each step. In the extreme cases where $q = 1$ or 0, the two particle columns are shifted intact to the right or left by one spatial interval, Δx, in each step.

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