LTCC Advanced Mathematics Series - Volume 2

Fluid and Solid Mechanics

Shaun Bullett
Tom Fearn
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editors

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Fluid and Solid Mechanics

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edited by Shaun Bullett, Tom Fearn & Frank Smith

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edited by Shaun Bullett, Tom Fearn & Frank Smith

Preface

The London Taught Course Centre (LTCC) for PhD students in the Mathematical Sciences has the objective of introducing research students to a broad range of topics. For some students, some of these topics might be of obvious relevance to their PhD projects, but the relevance of most will be much less obvious or apparently non-existent. However, all of us involved in mathematical research have experienced that extraordinary moment when the penny drops and some tiny gem of information from outside ones immediate research field turns out to be the key to unravelling a seemingly insoluble problem, or to opening up a new vista of mathematical structure. By offering our students advanced introductions to a range of different areas of mathematics, we hope to open their eyes to new possibilities that they might not otherwise encounter.

Each volume in this series consists of chapters on a group of related themes, based on modules taught at the LTCC by their authors. These modules were already short (five two-hour lectures) and in most cases the lecture notes here are even shorter, covering perhaps three-quarters of the content of the original LTCC course. This brevity was quite deliberate on the part of the editors — we asked the authors to confine themselves to around 35 pages in each chapter, in order to allow as many topics as possible to be included in each volume, while keeping the volumes digestible. The chapters are "advanced introductions", and readers who wish to learn more are encouraged to continue elsewhere. There has been no attempt to make the coverage of topics comprehensive. That would be impossible in any case — any book or series of books which included all that a PhD student in mathematics might need to know would be so large as to be totally unreadable. Instead what we present in this series is a cross-section of some of the topics, both classical and new, that have appeared in LTCC modules in the nine years since it was founded.

The present volume is within the area of fluid and solid mechanics. The main readers are likely to be graduate students and more experienced researchers in the mathematical sciences, looking for introductions to areas with which they are unfamiliar. The mathematics presented is intended to be accessible to first year PhD students, whatever their specialised areas of research. Whatever your mathematical background, we encourage you to dive in, and we hope that you will enjoy the experience of widening your mathematical knowledge by reading these concise introductory accounts written by experts at the forefront of current research.

Shaun Bullett, Tom Fearn, Frank Smith

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Chapter 1

Introductory Geophysical Fluid Dynamics

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This chapter concerns mathematical modelling of large-scale fluid flows relative to a rotating frame of reference, for which the effects of rotation are dominant and to leading order there is a balance of horizontal pressure gradients and Coriolis forces. The principal application is to oceanic and atmospheric flows with horizontal scales of tens of kilometres or more, and timescales of days or more. A fundamental equation in the dynamics of such flows is that for quasigeostrophic potential vorticity, and this is derived in the first part of the chapter, with stratification effects included in the form of layers with constant density within each layer. Large-scale wave-like behaviour is supported in the form of Rossby waves, and some basic properties of these waves are presented. Simplified conceptual quasigeostrophic models provide understanding of dynamical processes, and two examples are described: ocean spin-up and multiple equilibria.

1. Introduction

Mathematical representation of large-scale atmospheric and oceanic flows has great practical importance as it provides the basis for the dynamical numerical models used for making weather and climate outlooks for hours to decades ahead. The full equations of fluid motion are too complex to use for this purpose, but mathematical theory provides the foundation for approximations that represent the scales and phenomena of interest and allow efficient numerical computation. Even with these approximations, atmospheric and oceanic flows contain processes and interactions on a wide range of space and time scales. Mathematical models can further be used to focus on particular processes and investigate their behaviour and roles.

This chapter contains a subset of a course intended for graduates who are familar with the basics of fluid mechanics, such as the Navier–Stokes equations and wave-like behaviour such as gravity waves, but have not encountered geophysical fluid dynamics. A brief explanation of the governing equations for quasigeostrophic flow is provided, without rigorous justification for the various standard approximations employed.

Two examples are provided of conceptual models based on the quasigeostrophic potential vorticity equations. One is a model of mid-latitude wind-driven ocean circulation. The classic steady case demonstrates how intense currents such as the Gulf Stream occur near western boundaries, while the time-dependent part illustrates how Rossby waves influence ocean circulation and how in a stratified ocean they provide oceans with a long-term "memory". The second example demonstrates how the interaction of Rossby waves, topographic drag and mean flow may create multiple stable states, relevant in particular to "blocked" flow regimes in the atmosphere.

There are many good textbooks on this subject. More detailed and rigorous derivations of sets of equations relevant to geophysical fluid dynamics, with applications, may be found in books by Gill,[1] Pedlosky[2] and Vallis[3] for example.

2. Governing Equations

For flows relative to a rotating frame of reference, the Navier–Stokes equations have the form

$$\frac{\mathrm{D}\underline{u}}{\mathrm{D}t} + 2\underline{\Omega} \times \underline{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{u} + \text{gravitational effects}, \qquad (1)$$

where $\underline{u}$ is the velocity vector, p is pressure, ρ is density, $\mathrm{D}/\mathrm{D}t$ indicates a derivative following the motion and ν is a viscosity coefficient.

For planet Earth, the rotation vector $\underline{\Omega}$ has magnitude $\Omega = 2\pi$ radians per day and direction outward from the North Pole. Earth can be regarded as a sphere of radius R_e, with the atmospheric and oceanic flows in thin layers near that radius, with large horizontal scale compared to the depth in each medium.

With flows in mid-latitude regions in mind, choose a coordinate system that is centred on some latitude θ_0. For simplicity, locally Cartesian coordinates are defined, with x in the zonal (west to east) direction, y in the meridional (south to north) direction, and z vertically upwards. The corresponding fluid velocity components are denoted u (zonal), v (meridional) and w (upward). Note that $y = R_e(\theta - \theta_0)$, and at latitude θ the radially outward component of the rotation vector has size $\Omega \sin\theta$.

The horizontal momentum equations are

$$u_t + (\underline{u} \cdot \nabla)u - fv = -\frac{p_x}{\rho} + \nu\nabla^2 u, \tag{2a}$$

$$v_t + (\underline{u} \cdot \nabla)v + fu = -\frac{p_y}{\rho} + \nu\nabla^2 v. \tag{2b}$$

Here, $f = 2\Omega \sin\theta$ is referred to as the Coriolis parameter. Other components of the rotation vector other than f have small influence, and have been omitted in (2). (For brevity formal justifications of the approximations made here and elsewhere are omitted: details can be found in textbooks such as those cited in the introduction.)

The variation of f with latitude is fundamental to many properties of the large-scale flows of interest here. A further very common simplification is to consider f as a linear function of y, by using $\sin\theta \approx \sin\theta_0 + (\theta - \theta_0)\cos\theta_0$. Then $f \approx f_0 + \beta y$, where $f_0 = 2\Omega \sin\theta_0$ and $\beta = 2\Omega \cos\theta_0 / R_e$. With this assumption, the coordinate system is known as a "beta-plane". In standard terminology, a system with $\beta = 0$ is referred to as an "f-plane".

Mass conservation requires

$$\rho_t + \nabla \cdot (\rho \underline{u}) = 0. \tag{3}$$

2.1. *Hydrostatic balance*

For oceanic and atmospheric flows with horizontal scale much larger than the vertical scale, the vertical equation of motion is dominated by the balance of vertical pressure gradient and gravitational force. To a very good approximation, the system is in hydrostatic balance, with

$$p_z = -\rho g. \tag{4}$$

Effectively the pressure at any point is determined by the mass of overlying fluid, and is not influenced by the fluid motion.

2.2. *Geostrophic balance*

Suppose the flow has length scale L, a horizontal velocity scale U and an advective time scale L/U. The non-dimensional Rossby number R, fundamental to rotating flows, is defined as

$$R = \frac{U}{(f_0 L)}. \tag{5}$$

We assume the Rossby number is small ($R \ll 1$), in which case the left-hand side of (2) is dominated by the Coriolis terms fv and fu. Apart from thin layers near boundaries, viscous and forcing effects are small. The dominant balance is between the Coriolis and pressure gradient terms:

$$-fv = -\frac{p_x}{\rho}, \quad fu = -\frac{p_y}{\rho}. \tag{6}$$

This is referred to as "geostrophic balance". For later use, define a geostrophic horizontal flow u_g, v_g by

$$-v_g = -\frac{p_x}{(\rho_0 f_0)}, \quad u_g = -\frac{p_y}{(\rho_0 f_0)}, \tag{7}$$

where ρ_0 is a typical density scale. Note that $\underline{u}_g \cdot \nabla p = 0$: the geostrophic flow follows lines of constant pressure, i.e., isobars. Flow is cyclonic around low pressure centres, and anti-cyclonic around high pressure centres. In the northern hemisphere, orientation is such that cyclonic flow is anti-clockwise. Note also that $u_{gx} + v_{gy} = 0$, so the geostrophic flow is horizontally non-divergent. Thus a geostrophic streamfunction ψ can be defined: conventionally such that

$$u_g = -\psi_y, \quad v_g = \psi_x. \tag{8}$$

(Note: from here on assume ∇ is the horizontal gradient operator, and $\underline{u} = (u, v)$, as should be obvious from the context.)

2.3. *Representation of the density distribution as layers*

In practice, density varies throughout the ocean and atmosphere. The ideal gas law applies to the atmosphere, and in the ocean density depends mainly on temperature and salinity. As is often done for conceptual models and theoretical investigations, we will consider a simple representation of the density structure as layers in which each layer has a constant prescribed density. (See textbooks for alternative representations.) Thus, thermodynamic aspects that influence the density are not considered here: instead, the focus is on dynamic processes. We shall also assume that density varies little throughout the system. This is a viewpoint more directly suited to the oceans than the atmosphere: for the latter an alternative formulation can be made that allows for the substantial vertical variation of air density through the depth of the atmosphere at rest.

Suppose a system has N layers, with layer 1 at the top overlying layer 2 overlying layer 3, etc. Suppose layer n has density ρ_n, with $\rho_n < \rho_{n+1}$. With the fluid at rest the interfaces are horizontal, and the layers have depths H_n which are constant, except for the lowest layer whose depth may vary with x and y to allow the possibility of underlying topography. Suppose the top of layer 1 is at $z = z_T$ when undisturbed, and suppose the perturbation of this surface is η_1, so the top of disturbed layer 1 is at $z_T + \eta_1$. Similarly, η_2 denotes the perturbation to the interface between layers 1 and 2, so the bottom of layer 1 (and the top of layer 2) is at $z_T - H_1 + \eta_2$, and so on. A fixed perturbation $\eta_{N+1}(x, y)$ at the base of layer N can be used to represent the topography, so the bottom of layer N is at $z_T - (H_1 + \cdots + H_N) + \eta_{N+1}$. The density structure for such a system with two layers is illustrated in Fig. 1.

Using hydrostatic balance, some useful relationships between pressures and layer distributions can be derived. Suppose the pressure at the top surface is p_T at $z = z_T$. From the hydrostatic relation, in layer 1

$$p_1 = p_T + \rho_1 g \eta_1 - \rho_1 g (z - z_T) \tag{9}$$

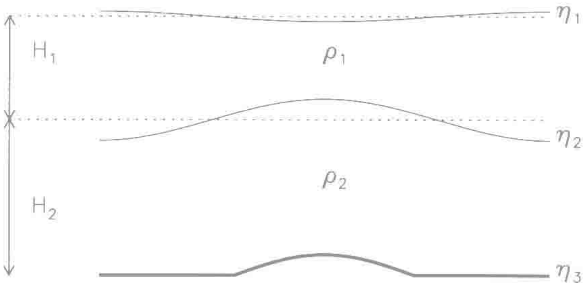

Fig. 1. Schematic diagram of the density structure in a two-layer system.

for $z_T - H_1 + \eta_2 < z < z_T + \eta_1$, and in layer 2

$$p_2 = p_T + \rho_1 g\eta_1 + (\rho_1 - \rho_2)g(H_1 - \eta_2) - \rho_2 g(z - z_T) \qquad (10)$$

for $z_T - H_1 - H_2 + \eta_3 < z < z_T - H_1 + \eta_2$, etc.

Suppose p_T is constant. (For oceanic applications, p_T is the sea level atmospheric pressure, fluctuations in which are small compared to pressure fluctuations below the surface and negligible for most circumstances. For atmospheric applications, p_T is effectively zero at the top of the atmosphere.) Then the horizontal pressure gradients are independent of depth within each layer:

$$\nabla p_1 = \rho_1 g\nabla\eta_1, \quad \nabla p_2 = \rho_1 g\nabla\eta_1 + (\rho_2 - \rho_1)g\nabla\eta_2, \text{ etc.} \qquad (11)$$

Thus, the geostrophic flow $\underline{u}_{g1}$ in layer 1, determined by ∇p_1, can be diagnosed from η_1 and is independent of depth in layer 1; likewise $\underline{u}_{g2}$ is determined by η_1 and η_2, etc. (Thus $\underline{u}_g$ is determined by the overlying density structure.) Note that

$$\nabla(p_2 - p_1) = (\rho_2 - \rho_1)g\nabla\eta_2, \qquad (12)$$

so the difference $\underline{u}_{g1} - \underline{u}_{g2}$ is determined by η_2. The vertical geostrophic shear between two layers is determined by the horizontal gradient in the intervening density structure.

With density constant within each layer, from (3) it follows that

$$u_x + v_y + w_z = 0. \qquad (13)$$

2.4. *Shallow water equations and potential vorticity*

The above relations show how the density structure, pressure gradients and geostrophic flows are diagnostically related. (In particular, given all the ρ_n and η_n, $\underline{u}_g$ are known.) However, more information is needed to find out how they evolve. The relevant equations can be derived rigorously through asymptotic expansions with the Rossby number as a small parameter. Here, a more *ad hoc* approach is adopted, making a series of assumptions that can ultimately be justified more formally.

Within each layer (away from possible thin frictional layers at the boundaries) the horizontal flow is independent of depth and governed by the "shallow water equations":

$$u_t + (\underline{u} \cdot \nabla)u - fv = -\frac{p_x}{\rho_0} + A\nabla^2 u, \tag{14a}$$

$$v_t + (\underline{u} \cdot \nabla)v + fu = -\frac{p_y}{\rho_0} + A\nabla^2 v. \tag{14b}$$

The situation is analogous to long waves (wavelength much greater than depth) in shallow water, hence the name. Note that a constant reference density is used in the pressure gradient terms, valid for all layers. Here, A is a weak horizontal diffusivity coefficient, whose influence is negligible except in regions of strong gradients. (This diffusivity is intended to represent the effects of small scales for which the approximate equations are no longer valid, rather than molecular viscosity.)

The vorticity (more correctly, the vertical component of the vorticity vector) is $\zeta = v_x - u_y$. The vector identity $(\underline{u} \cdot \nabla)\underline{u} = \nabla \underline{u}^2/2 - \underline{u} \times \zeta\underline{k}$, where $\underline{k}$ is the unit vertical vector, can be used to write (14) as

$$u_t + \frac{1}{2}(\underline{u}^2)_x - (f + \zeta)v = -\frac{p_x}{\rho_0} + A\nabla^2 u, \tag{15a}$$

$$v_t + \frac{1}{2}(\underline{u}^2)_y + (f + \zeta)u = -\frac{p_y}{\rho_0} + A\nabla^2 v. \tag{15b}$$

Eliminating p and $\underline{u}^2$ by cross-differentiating leads to the vorticity equation

$$\zeta_t + \underline{u} \cdot \nabla(\zeta + f) + (\zeta + f)\nabla \cdot \underline{u} = A\nabla^2 \zeta. \tag{16}$$

From (13), it follows that

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} = (\zeta + f)w_z + A\nabla^2\zeta, \tag{17}$$

where the material derivative is $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \underline{u} \cdot \nabla$. Thus changes in the total vorticity $\zeta + f$ are induced by vertical motion stretching or shrinking the fluid column within the layer and by dissipation.

As u and v are independent of depth within the layer, it follows from (13) that w_z is depth-independent and hence

$$w_z = \frac{(w_T - w_B)}{h}, \tag{18}$$

where h denotes the layer depth and w_T and w_B denote w near the top and bottom of the layer.

The vertical motions w_T and w_B are influenced both by the upper and lower layer boundary positions and by thin frictional layers known as Ekman layers. Within these thin boundary layers, the flow is adjusted to match boundary conditions at the layer boundaries where necessary. For brevity, the standard properties of Ekman layers are not derived here, but will simply be stated as required in later sections. For now, write

$$w_T - w_B = \frac{\mathrm{D}h}{\mathrm{D}t} + w_{ET} - w_{EB}, \tag{19}$$

indicating the contributions from layer boundary positions and from the thin Ekman layers.

Noting that

$$\frac{D}{Dt}\frac{(\zeta + f)}{h} = \frac{1}{h}\frac{D}{Dt}(\zeta + f) - \frac{(\zeta + f)}{h^2}\frac{D}{Dt}h,$$

it follows from (17) and (18) that

$$\frac{D}{Dt}\frac{(\zeta + f)}{h} = \frac{(\zeta + f)}{h}\frac{(w_{ET} - w_{EB})}{h} + \frac{1}{h}A\nabla^2\zeta. \tag{20}$$

The expression $(\zeta + f)/h$ is the potential vorticity, which is a fundamental quantity in geophysical fluid dynamics. (There are equivalent expressions in continuously stratified systems.)