Theory of Stabilization for Linear Boundary Control Systems

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To my family,
Mariko, Ryutaro, and Hiromu.

Preface

This monograph studies the stabilization theory for linear systems governed by partial differential equations of parabolic type in a unified manner. As long as controlled plants are relatively small, such as electric circuits and mechanical oscillations/rotations of rigid bodies, ordinary differential equations, abbreviated as *ode(s)*, are suitable mathematical models to describe them. When the controlled plants are, e.g., chemical reactors, wings of aircrafts, or other flexible systems such as robotics arms, plates, bridges, and cranes, however, effects of space variables are essential and non-neglegeble terms. For the set up of mathematical models describing these plants, partial differential equations, abbreviated as *pde(s)*, are a more suitable language. It is generally expected that control laws based on more accurate pde models would work effectively in actual applications.

The origin of control theory is said to be the paper, "On governors" by J.C. Maxwell (1868). For many years, control theory has been studied mainly for systems governed by odes in which controlled plants are relatively small. Control theory for pdes began in 60's of the 20th century, and the study of stabilization in mid 70's to cope with much larger systems. Fundamental concepts of control such as controllability, observability, optimality, and stabilizability are the same as in those of odes, and translated by the language of pdes. The essence of pdes consists in their infinite-dimensional properties, so that control problems of pdes face serious difficulties in respective aspects, which have never been experienced in the world of odes: However, these difficulties provide us rich and challenging fields of study both from mathematical and engineering viewpoints.

Among other control problems of pdes such as optimal control problems, etc., we concentrate ourselves on the topic of stabilization problems. Stabilization problems of pdes have a new aspect of pdes in the framework of *synthesis* (or design) of a desirable spectrum by involving the concept of observation/control, and are connected not only with functional analysis but also non-harmonic analysis and classical Fourier analysis, etc. The monograph consists of eight chapters which strongly reflects the author's works over thirty years except for Chapter 2: Some were taught in graduate courses at Kobe University. The organization of the monograph is stated as follows: It begins with the linear tabilization problem of finite dimension in Chapter 1. Finite-dimensional models constitute *pseudo*-internal structures of pdes. Although the problem is entirely solved by W. M. Wonham in 1967 [70], we develop a much easier new approach, which has never appeared even among the community of finite-dimensional control theory: It is based on Sylvester's equation. Infinite-dimensional versions of the equation appear in later chapters as an essential tool for stabilization problems throughout the monograph. Chapter 2 is a brief introduction of basic results on standard elliptic differential operators L and related Sobolev spaces necessary for our control problems: These results are well known among the pdes community, but proofs of some results are stated for the readers' convenience. As for results requiring much preparation we only provide some references instead of proofs. In Chapters 3 through 7, the main topics discussed are, where stabilization problems of linear parabolic systems are successfully solved in the boundary observation/boundary feedback scheme. The elliptic operator L is derived from a pair of standard (but general enough) differential operators $(\mathscr{L}, \tau)$, and forms the coefficient of our control systems, where $\mathscr{L}$ denotes a uniformly elliptic differential operator and τ a boundary operator. The operator L is sectorial, and thus $-L$ turns out to be an infinitesimal generator of an analytic semigroup. One of important issues is certainly the existence or non-existence of Riesz bases associated with L: When an associated Riesz basis exists, a sequence of finite-dimensional approximation models of the original pde is quantitatively justified, so that the control laws based on the approximated finite-dimensional models effectively works. There is an attempt to draw out a class of elliptic operators with Riesz bases (see the footnote in the beginning of Chapter 4). However, is the class of pdes admitting associated Riesz bases general enough or much narrower than expected? We do not have a satisfactory solution to the question yet. Based on these observations, our feedback laws are constructed so that they are applied to a general class of pdes, without assuming Riesz bases.

There are two kinds of feedback schemes: One is a *static feedback* scheme, and the other a *dynamic feedback* scheme. In Chapter 3, the stabilization problem and related problems are discussed in the static feedback scheme, in which the outputs of the system are directly fed back into the system through the actuators. While the scheme has difficulties in engineering implementations, it works as an auxiliary means in the dynamic feedback schemes. In Chapter 4, we establish stabilization in the scheme of boundary observation/boundary feedback. The feedback scheme is the dynamic feedback scheme, in which the outputs on the boundary are fed back into the system through another differential equation described in another abstract space. This differential equation is called a *dynamic compensator*, the concept of which originates from D. G. Luenberger's paper [33] in 1966 for linear odes. In his paper, two kinds of compensators are proposed: One is an *identity* compensator, and the other a compensator of *general type*. We formulate the latter compensator in the feedback loop to cope with the stabilization problem, and finally reduce the compensator to a finite-dimensional one. All arguments are algebraic, and do not depend on the kind of boundary operators τ. In Chapter 5, the problem is discussed from another viewpoint when the system admits a Riesz basis. Since a finite-dimensional approximation to the pde is available as a strongly effective means, an identity compensator is installed in the feedback loop. Most stabilization results in the literature are based on identity compensators, but have difficulty in terms of mathematical generality. In Chapters 4 and 5, observability and controllability conditions on sensors and actuators, respectively, are assumed on the *pseudo*-internal substructure of finite dimension. We then ask in Chapter 6 the following: What can we claim when the observability and controllability conditions are lost? *Output stabilization* is one of the answers: Assuming an associated Riesz basis, we propose sufficient conditions on output stabilization. A related problem is also discussed, which leads to a new problem, that is, the problem of *pole allocation with constraints*. To show mathematical generality of our stabilization scheme, we generalize in Chapter 7 the class of operators L, in which $-L$ is a generator of *eventually differentiable* semigroups: A class of delay-differential equations generates such operators L.

In our general stabilization scheme, we solve an inverse problem associated with the infinite-dimensional Sylvester's equation. The problem forms a so called *ill-posed* problem lacking of continuity property. Finally in Chapter 8, we propose a numerical approximation algorhism to the inverse problem, the solution of which is mathematically ensured. The algorhism consists of a simple idea, but needs tedious calculations. Although the algorhism has some restrictions at present, it is expected that it would work in more general settings of the parameters. Numerical approximation itself is a problem independent of our stabilization problem. However, the latter certainly leads to a development of new problems in numerical analysis. The author hopes that willing readers could open a new area in effective numerical algorhisms.

The author in his graduate school days had an opportunity to read papers by Y. Sakawa, by H. O. Fattorini, and by S. Agmon and L. Nirenberg ([2, 17, 18, 57]) among others, and learned about the close relationships lying in differential equations, functional analysis, and the theory of functions. Inspired by these results, he had a hope to contribute to deep results of such nature, since then. He is not certain now, but would be happy, if the monograph coould reflect his hope even a little.

Takao Nambu

December, 2015
Kobe

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Chapter 1

Preliminary results - Stabilization of linear systems of finite dimension

1.1 Introduction

We develop in this chapter the basic problem arising from stabilization problems of finite-dimension. Since the celebrated pole assignment theory [70] (see also [56, 68]) for linear control systems of finite dimension appeared, the theory has been applied to various stabilization problems both of finite dimension and infinite dimension such as the one with boundary output/boundary input scheme (see, e.g., [12, 13, 28, 37 – 40, 42 – 45, 47 – 50, 53, 58, 59] and the references therein). The symbol H_n, $n = 1, 2, \ldots$, hereafter will denote a finite-dimensional Hilbert space with $\dim H_n = n$, equipped with inner product $\langle \cdot, \cdot \rangle_n$ and norm $\|\cdot\|$. The symbol $\|\cdot\|$ is also used for the $\mathscr{L}(H_n)$-norm. Let L, G, and W be operators in $\mathscr{L}(H_n)$, $\mathscr{L}(\mathbb{C}^N; H_n)$, and $\mathscr{L}(H_n; \mathbb{C}^N)$, respectively. Here and hereafter, the symbol $\mathscr{L}(R; S)$, R and S being linear spaces of finite or infinite dimension, means the set of all linear bounded operators mapping R into S. The set $\mathscr{L}(R; S)$ forms a linear space. When $R = S$, $\mathscr{L}(R; R)$ is abbreviated simply as $\mathscr{L}(R)$. Given L, W, and any set of n complex numbers, $Z = \{\zeta_i\}_{1 \leqslant i \leqslant n}$, the problem is to seek a suitable G such that $\sigma(L - GW) = Z$, where $\sigma(L - GW)$ means the spectrum of the operator

$L - GW$. Or alternatively, given L and G, its algebraic counterpart is to seek a W such that $\sigma(L - GW) = Z$. Stimulated by the result of [70], various approaches and algorhisms for computation of G or W have been proposed since then (see, e.g., [7, 10, 14]). However, each approach needs much preparation and a deep background in linear algebra to achieve stabilization and determine the necessary parameters. Explicit realizations of G or W sometimes seem complicated. One for this is no doubt the complexity of the process in determining G or W *exactly* satisfying the relation, $\sigma(L - GW) = Z$.

Let us describe our control system: Our system, consisting of a state $u(\cdot) \in H_n$, output $y = Wu \in \mathbb{C}^N$, and input $f \in \mathbb{C}^N$, is described by a linear differential equation in H_n,

$$\frac{du}{dt} + Lu = Gf, \quad y = Wu, \quad u(0) = u_0 \in H_n. \tag{1.1}$$

Here,

$$\begin{aligned} Gf &= \sum_{k=1}^{N} f_k g_k \quad \text{for } f = (f_1 \; \dots \; f_N)^{\mathrm{T}} \in \mathbb{C}^N, \\ Wu &= \left(\langle u, w_1 \rangle_n \quad \dots \quad \langle u, w_N \rangle_n\right)^{\mathrm{T}} \quad \text{for } u \in H_n, \end{aligned} \tag{1.2}$$

$(\dots)^{\mathrm{T}}$ denoting the transpose of vectors or matrices throughout the monogtaph. The vectors $w_k \in H_n$ denote given weights of the observation (output); and $g_k \in H_n$ are actuators to be constructed. By setting $f = y$ in (1.1), the control system yields a feedback system,

$$\frac{du}{dt} + (L - GW)u = 0, \quad u(0) = u_0 \in H_n. \tag{1.3}$$

According to the choice of a basis for H_n, the operators L, G, and W are identified with matrices of respective size. We hereafter employ the above symbols somewhat different from those familiar in the control theory community of finite dimension, in which state of the system, for example, would be often represented as $x(\cdot)$; output Cx; input u; and equation

$$\frac{dx}{dt} = Ax + Bu = (A + BC)x, \quad u = Cx.$$

The reason for employing present symbols is that they are consistent with those in systems of infinite dimension discussed in later chapters.

Let us assume that $\sigma(L) \cap \mathbb{C}_- \neq \varnothing$, so that the system (1.1) with $f = 0$ is unstable. Given a $\mu > 0$, the *stabilization problem* for the finite dimensional control system (1.3) is to seek a G or W such that

$$\left\| e^{-t(L-GW)} \right\| \leqslant \text{const } e^{-\mu t}, \quad t \geqslant 0. \tag{1.4}$$

The pole assignment theory [70] plays a fundamental role in the above problem, and has been applied so far to various linear systems. The theory is concretely stated as follows: *Let* $Z = \{\zeta_i\}_{1 \leqslant i \leqslant n}$ *be any set of n complex numbers, where some* ζ_i *may coincide. Then, there exists an operator G such that* $\sigma(L - GW) = Z$, *if and only if the pair* (W, L) *is observable*. Thus, if the set Z is chosen such that $\min_{\zeta \in Z} \mathrm{Re}\, \zeta$, say μ ($= \mathrm{Re}\, \zeta_1$) is positive, and if there is *no* generalized eigenspace of $L - GW$ corresponding to ζ_1, we obtain the decay estimate (1.4).

Now we ask: Do we need *all* information on $\sigma(L - GW)$ for stabilization? In fact, to obtain the decay estimate (1.4), it is not necessary to designate *all* elements of the set Z: What is really necessary is the number, $\mu = \min_{\zeta_i \in Z} \mathrm{Re}\, \zeta_i$, say $= \mathrm{Re}\, \zeta_1$, and the spectral property that ζ_1 does not allow any generalized eigenspace; the latter is the requirement that *no* factor of algebraic growth in time is added to the right-hand side of (1.4). In fact, when an algebraic growth is added, the decay property becomes a little worse, and the gain constant ($\geqslant 1$) in (1.4) increases. The above operator $L - GW$ also appears, as a *pseudo*-substructure, in the stabilization problems of infinite dimensional linear systems such as parabolic systems and/or retarded systems (see, e.g., [16]): These systems are decomposed into two, and understood as composite systems consisting of two states; one belonging to a finite dimensional subspace, and the other to an infinite dimensional one. It is impossible, however, to manage the infinite dimensional substructures. Thus, no matter how *precisely* the finite dimensional spectrum $\sigma(L - GW)$ could be assigned, it does not exactly dominate the whole structure of infinite dimension. In other words, the assigned spectrum of finite dimension is not necessarily a subset of the spectrum of the infinite-dimensional feedback control system.

In view of the above observations, our aim in this chapter is to develop a new approach much simpler than those in existing literature, which allows us to construct a desired operator G or a set of actuators g_k ensuring the decay (1.4) in a simpler and more explicit manner (see (2.10) just below Lemma 2.2). The result is, however, not as sharp as in [70] in the sense that it does not generally provide the precise location of the assigned eigenvalues. From the above viewpoint of infinite-dimensional control theory, however, the result would be meaningful enough, and satisfactory for stabilization. We note that our result exactly coincides with the standard pole assignment theory in the case where we can choose $N = 1$ (see Proposition 2.3 in Section 2). The results of this chapter are based on those discussed in [48, 51, 52].

Our approach is based on Sylvester's equation of finite dimension. Sylvester's equation in infinite-dimensional spaces has also been studied extensively (see, e.g., [6] for equations involving only bounded operators), and even the unboundedness of the given operators are allowed [37, 39, 40, 42 – 45, 47, 49, 50, 53]. Sylvester's equation in this chapter is of finite dimension, so that there arises no difficulty caused by the complexity of infinite dimension. Its infinite-dimensional version and the properties are discussed later in Chapters 4, 6, and 7. Given a positive integer s and vectors $\xi_k \in H_s$, $1 \leqslant k \leqslant N$, let us consider the following Sylvester's equation in H_n:

$$XL - MX = -\Xi W, \quad \Xi \in \mathscr{L}(\mathbb{C}^N; H_s), \quad \text{where}$$
$$\Xi z = \sum_{k=1}^{N} z_k \xi_k \quad \text{for } z = (z_1 \ \dots \ z_N)^{\mathrm{T}} \in \mathbb{C}^N. \tag{1.5}$$

Here, M denotes a given operator in $\mathscr{L}(H_s)$, and ξ_k vectors to be designed in H_s. A possible solution X would belong to $\mathscr{L}(H_n; H_s)$. An approach via Sylvester's equations is found, e.g., in [7, 10], in which, by setting $n = s$, a condition for the existence of the bounded inverse $X^{-1} \in \mathscr{L}(H_n)$ is sought. Choosing an M such that $\sigma(M) \subset \mathbb{C}_+$, it is then proved that

$$L + (X^{-1}\Xi)W = X^{-1}MX, \quad \sigma(X^{-1}MX) = \sigma(M) \subset \mathbb{C}_+,$$

the left-hand side of which means a desired perturbed operator. The procedure of its derivation is, however, rather complicated, and the choice of the ξ_k is unclear. In fact, X^{-1} might not exist sometimes for some ξ_k.

The approach in this chapter is new and rather different. Let us characterize the operator L in (1.5). There is a set of generalized eigenpairs $\{\lambda_i, \varphi_{ij}\}$ with the following properties:

(i) $\sigma(L) = \{\lambda_i;\ 1 \leqslant i \leqslant \nu\,(\leqslant n)\}$, $\quad \lambda_i \neq \lambda_j$ for $i \neq j$; and

(ii) $L\varphi_{ij} = \lambda_i \varphi_{ij} + \sum_{k<j} \alpha^i_{jk} \varphi_{ik}$, $\quad 1 \leqslant i \leqslant \nu$, $1 \leqslant j \leqslant m_i$.

Let P_{λ_i} be the projector in H_n corresponding to the eigenvalue λ_i. Then, we see that $P_{\lambda_i} u = \sum_{j=1}^{m_i} u_{ij} \varphi_{ij}$ for $u \in H_n$. The restriction of L onto the invariant subspace $P_{\lambda_i} H_n$ is, in the basis $\{\varphi_{i1}, \dots, \varphi_{im_i}\}$, is represented by the $m_i \times m_i$ upper triangular matrix Λ_i, where

$$\Lambda_i|_{(j,k)} = \begin{cases} \alpha^i_{kj}, & j < k, \\ \lambda_i, & j = k, \\ 0, & j > k. \end{cases} \tag{1.6}$$

If we set $\Lambda_i = \lambda_i + N_i$, the matrix N_i is nilpotent, that is, $N_i^{m_i} = 0$. The minimum integer n such that $\ker N_i^n = \ker N_i^{n+1}$, denoted as l_i, is called the *ascent* of $\lambda_i - L$. It is well known that the ascent l_i coincides with the order of the pole λ_i of the resolvent $(\lambda - L)^{-1}$. Laurent's expansion of $(\lambda - L)^{-1}$ in a neighborhood of the pole $\lambda_i \in \sigma(L)$ is expressed as

$$(\lambda - L)^{-1} = \sum_{j=1}^{l_i} \frac{K_{-j}}{(\lambda - \lambda_i)^j} + \sum_{j=0}^{\infty} (\lambda - \lambda_i)^j K_j, \quad \text{where}$$
$$l_i \leqslant m_i, \quad K_j = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - \lambda_i| = \delta} \frac{(\zeta - L)^{-1}}{(\zeta - \lambda_i)^{j+1}}\, d\zeta, \quad j = 0, \pm 1, \pm 2, \dots. \tag{1.7}$$