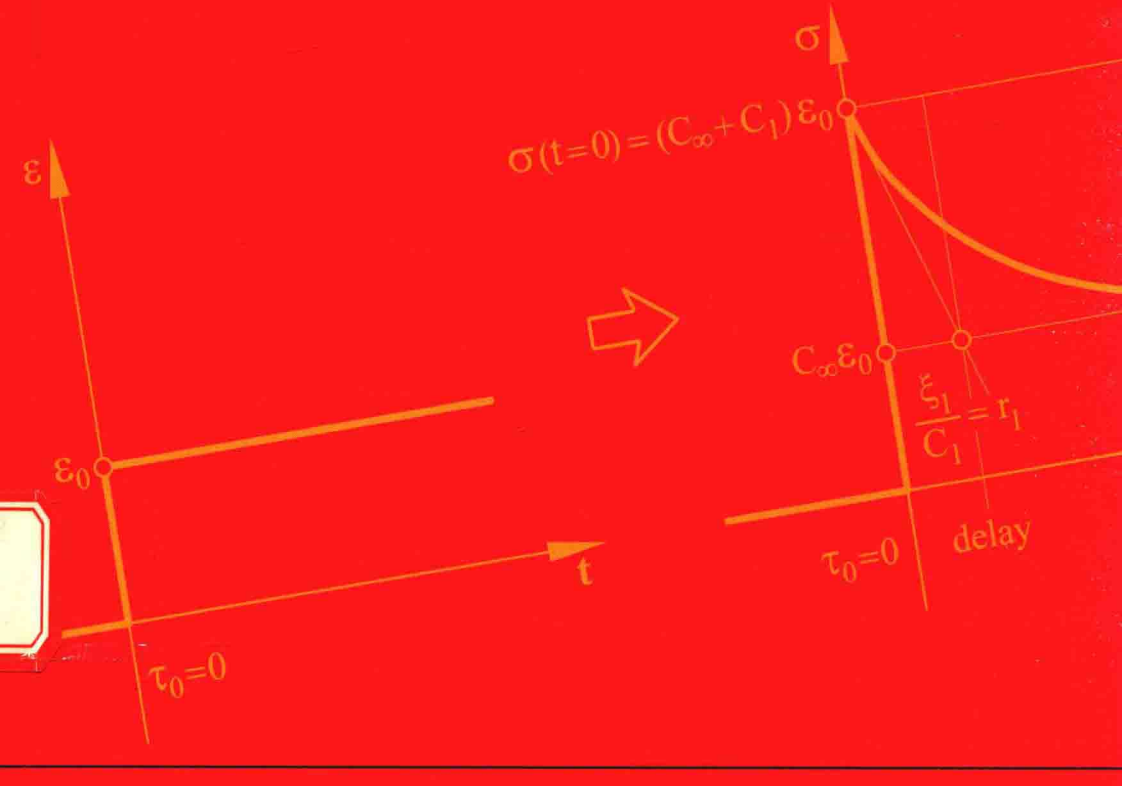
Nonlinear Dynamics of Structures

Sergio Oller



Lecture Notes
on Numerical Methods
in Engineering and Sciences

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Lecture Notes on Numerical Methods in Engineering and Sciences

Aims and Scope of the Series

This series publishes text books on topics of general interest in the field of computational engineering sciences.

The books will focus on subjects in which numerical methods play a fundamental role for solving problems in engineering and applied sciences. Advances in finite element, finite volume, finite differences, discrete and particle methods and their applications are examples of the topics covered by the series.

The main intended audience is the first year graduate student. Some books define the current state of a field to a highly specialised readership; others are accessible to final year undergraduates, but essentially the emphasis is on accessibility and clarity.

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This work is dedicated to my wife and son, and also to all my loved ones

Preface

This book has been written to present the conceptual basis of "Nonlinear Dynamics" of structural systems. Although there are many papers on this subject, I have decided to write this book for educational purposes addressed to students with an academic level equivalent to a master's degree.

The book is divided into three main parts: the first one sets up the basis on which the nonlinear dynamics applied to discrete structures is based on; the second one shows the effect of time-independent constitutive model behavior within the nonlinear dynamic response; and finally, the third part analyzes the effect of time-dependent constitutive models in a nonlinear dynamic behavior.

This work has been possible thanks to the institutional support of CIMNE (International Center for Numerical Methods in Engineering), which has financially supported this book since its first edition in Spanish in 2002, and later in its English edition. Many people have participated in the latter, and I would particularly like to thank Ms. Hamdy Briceño, Prof. Miguel Cerrolaza and Cristina Pérez Arias for their careful translation and revision of this text. I would also like to thank all my students who have contributed to the correction of the text during the eleven years that this book has been used as a syllabus of the "Nonlinear Dynamics" course in the Department of Strength of Materials, at the Technical University of Catalonia, Spain.

I hope these notes will contribute to a better understanding of the nonlinear dynamics and encourage the reader to study this subject in greater depth.

Barcelona, May 2014

Sergio Oller

Contents

1 Introduction

Structural dynamics studies the structural equilibrium over time among external forces, elastic forces, mass forces and viscous forces for a discrete structural system with points that are internally linked to each other and all linked to a fixed reference system. These internal links between points describing the structural system may be elastic or not. If they are not elastic, the behavior of the system of points is non-conservative and therefore the structural material has a *nonlinear dissipative constitutive behavior.* Additionally to this nonlinear behavior, there is also a *nonlinear dissipative behavior due to the effects of the material viscosity* that leads to viscous forces dependent on the system velocity. In simpler cases, the damping non linearity is due to the development of viscous forces proportional to the velocity; however, in more complex cases the viscosity term may be time-dependent. Also, the system's non linearity can be observed in systems having large displacements and where the system works beyond its original geometric configuration, leading to a *nonlinear kinematic behavior.* Such non linearity is even more pronounced when large strain occurs along with large displacements, turning the solution of the structure's dynamic problem more complex.

All the above mentioned subjects will be thoroughly studied in this work; concepts are based on the *nonlinear dynamics of structures,* on the *mechanics of continuum media* and on *numerical techniques* such as the *finite element method.*

A nonlinear structural dynamics course may have different approaches to the content and development of concepts it must have and all of them are valid as long as the goals are achieved. This work deals with the required concepts to complete the basic training in structural nonlinear dynamics, in the mechanics of continuum media and in the finite element method. Accordingly, the topics included in this basic training in structures that are assumed to be already known by the reader will not be studied again.

A brief description of the book's contents follows: in **chapter 2**, an *introduction to the thermodynamical basis of the motion equation* is presented. This fundamental chapter contains the origins of the problem, which is set within a structured formulation that can address all remaining items in a consistent way. In **chapter 3**, the methods to solve the motion equation are described in detail; both the implicit and the explicit procedures and the advantages and drawbacks of each method are analyzed. In **chapter 4**, the stability concept of the solution of conservative systems is studied for different methods in order to solve the equation of movement. Once the basis of the solution stability of linear systems are established, an approximation to the nonlinear problem is made and criteria for the stability study are provided. Energy conservation here is a crucial requirement. This leads to the "formulation of conservative solution methods" currently being used in nonlinear dynamics. In **chapter 5,** once the basis of nonlinear dynamics are set, the time-independent constitutive formulation, such as plasticity and damage, is addressed showing also how the structural nonlinearity is affected by these behaviors. Similarly, in **chapter 6** the constitutive behavior of time-dependent materials, such as delayed elasticity and relaxation, is detailed, where the nonlinear damping is included in a natural way. This is emphasized because it is considered a part of the nonlinear dynamics where there is a conceptual gap.

2 Thermodynamic Basis of the Equation of Motion

2.1 Introduction

The thermodynamic basis defining the linear or nonlinear behavior of a solid during the mechanical process is introduced in this chapter. The synthesized concepts here help to understand the solid nonlinear behavior and to clearly set equilibrium at every time.

The kinematics of deformable solids is briefly reviewed to establish the notation to be used as well as the definitions of the mechanics of continuum media which are important to remember. A brief description of the *thermodynamics* is also presented to point out the most relevant aspects of the formulation of constitutive models for the nonlinear behavior of solids. Reference to the mechanics of continuous media and to thermodynamics[1,2,3] is highly recommended to deepen and broaden the concepts addressed here.

2.2 Kinematics of deformable bodies

In order to sustain the formulation of constitutive models, it is necessary to introduce the basic concepts describing the kinematics of a point in the space, the stress and the strain measurements as well as their relation in different configurations. The purpose of this chapter is to establish the notation and review some definitions. It is not intended to substitute any specific book of continuum mechanics. Therefore, reference to the sources[1,2,3] is recommended.

2.2.1 Basic definitions of tensors describing the kinematics of a point in the space

Let a continuous solid in three dimensions be considered, represented by the domain $\Omega_t \subset \mathbb{R}^3$ located in the space in its *current configuration* in time t, or by an image of this domain located in the space in an *intermediate configuration* $\overline{\Omega}_t \subset \mathbb{R}^3$ or by the domain $\Omega_0 \subset \mathbb{R}^3$ located in the *reference configuration* or *original configuration* (see Figure 2.1).

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[1] Malvern, L. (1969). *Introduction to the mechanics of continuous medium.* Prentice Hall, Englewood Cliffs, NJ.
[2] Lubliner, J. (1990). *Plasticity theory.* MacMillan, New York.
[3] Maugin, G. A. (1992). *The thermomechanics of plasticity and fracture.* Cambridge University Press.

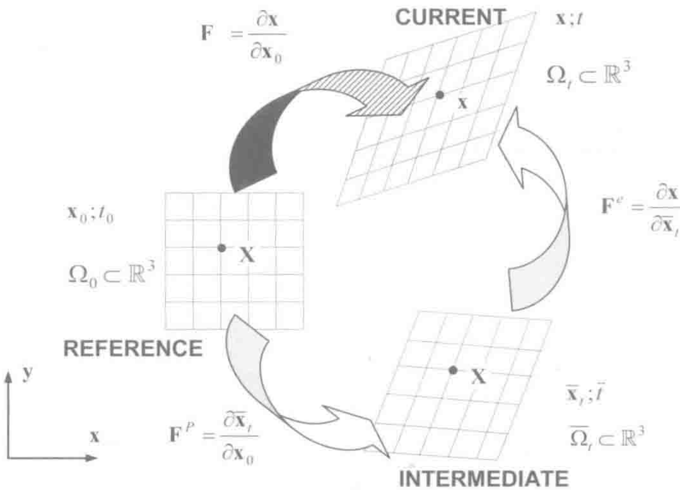

Figure 2.1 –Schematic representation of the kinematic configurations of a solid in the space.

A point $\mathbf{X} \in \Omega_0$, of coordinates $(\mathbf{x}_i)_0$, located at the *reference configuration*, to which one and only one of the points in the *intermediate configuration* corresponds, represented by $\mathbf{X} \in \overline{\Omega}_t$ of coordinates $(\overline{\mathbf{x}}_i)_t$, and similarly corresponds to it $\mathbf{X} \in \Omega_t$ with coordinates $(\mathbf{x}_i)_t$ corresponding to the *current configuration*. Thus, the body movement is described as a function of its position in the reference configuration and of the time,

$$\mathbf{x} = \mathbf{x}(\mathbf{X};t); \quad \mathbf{X} \in \Omega_0 \tag{2.1}$$

The *gradient of deformation tensor* is defined as the following transformation

$$\mathbf{F} = \mathbf{F}(\mathbf{X};t) = \nabla_0 \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \mathbf{J} \tag{2.2}$$

where $\mathbf{J}$ is *the Jacobian matrix*. The remaining transformations shown in Figure 2.1 are obtained from the following definition,

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \frac{\partial \mathbf{x}}{\partial \overline{\mathbf{x}}_t}\frac{\partial \overline{\mathbf{x}}_t}{\partial \mathbf{x}_0} = \mathbf{F}^e \cdot \mathbf{F}^p \tag{2.3}$$

where

$$\begin{aligned} \mathbf{F}^e &= \frac{\partial \mathbf{x}}{\partial \overline{\mathbf{x}}_t} \quad \text{Elastic transformation,} \\ \mathbf{F}^p &= \frac{\partial \overline{\mathbf{x}}_t}{\partial \mathbf{x}_0} \quad \text{Plastic transformation,} \end{aligned} \tag{2.4}$$

The change of the solid volume during a configuration change is obtained by the determinant of the Jacobian matrix commonly known as *Jacobian*. Thus,

$$J = |\mathbf{J}| = |\mathbf{F}| = \frac{dV}{dV_0} > 0 \tag{2.5}$$

dV and dV_0 are the infinitesimal volume in the configurations Ω_t and Ω_0, respectively.

The strain gradient tensor can be decomposed as the following polar transformation,

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \tag{2.6}$$

where $\mathbf{R}$ is the so called *orthogonal tensor*, which meets the following orthonormal condition $\mathbf{R} \cdot \mathbf{R}^T \equiv \mathbf{R}^T \cdot \mathbf{R} \equiv \mathbf{I}$, and both $\mathbf{U}$ and $\mathbf{V}$ are positive-defined symmetric tensors. The definition of the latter depends on the Cauchy-Green *right tensor* $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$, then the *right stretching tensor* is equal to $\mathbf{U} = \mathbf{C}^{1/2}$. The Cauchy Green *left tensor* is also defined as $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$, so that by substituting equation (2.6) into the latter $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T \equiv \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{U} \cdot \mathbf{R}^T = \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^T$ is then obtained and from here the *left stretching tensor* can also be defined as $\mathbf{V} = \mathbf{B}^{1/2}$, so it can be rewritten as $\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T = \mathbf{F} \cdot \mathbf{R}^T$. From here it is obvious that the gradient of deformation can also be written as $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$.

Following Noll's notation (see Lubliner[2]), $\vec{\mathcal{V}}_{\mathbf{x}}$ is called an Euclidian generic spatial vector defined in any configuration $\mathbf{x}$; thus, $\vec{\mathcal{V}}_0$ and $\vec{\mathcal{V}}$ will be the vectors defined in the *reference* and *current* configuration, respectively. The linear continuous space is designated as $L(\mathbf{x};\mathbf{y})$ that transforms $\mathbf{x} \to \mathbf{y}$. Based on this criterion, these tensors are designated according to the origin and destination of the transformation they perform.

$$\begin{array}{lll}
\mathbf{C} \in L\left(\vec{\mathcal{V}}_0;\vec{\mathcal{V}}_0\right) \quad ; & \mathbf{U} \in L\left(\vec{\mathcal{V}}_0;\vec{\mathcal{V}}_0\right) & : \text{Reference tensors - Lagrangeans,} \\
\mathbf{B} \in L\left(\vec{\mathcal{V}};\vec{\mathcal{V}}\right) \quad ; & \mathbf{V} \in L\left(\vec{\mathcal{V}};\vec{\mathcal{V}}\right) & : \text{Current Tensors - Eulerians,} \\
\mathbf{F} \in L\left(\vec{\mathcal{V}}_0;\vec{\mathcal{V}}\right) \quad ; & \mathbf{R} \in L\left(\vec{\mathcal{V}}_0;\vec{\mathcal{V}}\right) & \left.\right\} : \text{Bipunctual Tensors.} \\
\mathbf{F}^e \in L\left(\vec{\mathcal{V}}_{\mathrm{p}};\vec{\mathcal{V}}\right) \quad ; & \mathbf{F}^p \in L\left(\vec{\mathcal{V}}_0;\vec{\mathcal{V}}_{\mathrm{p}}\right) &
\end{array} \tag{2.7}$$

Tensors $\mathbf{F}^e \in L\left(\vec{\mathcal{V}}_{\mathrm{p}};\vec{\mathcal{V}}\right)$ and $\mathbf{F}^p \in L\left(\vec{\mathcal{V}}_0;\vec{\mathcal{V}}_{\mathrm{p}}\right)$ are also called *material tensors* and they are invariant under any Euclidian transformation.

2.2.2 Strain measurements

The strain in the reference configuration, also called *Lagrangian strain*, is defined as:

$$\mathbf{E}_n = \frac{1}{n}\left(\mathbf{U}^n - \mathbf{I}\right) \tag{2.8}$$

Then, the following strain measurements are obtained:

$$\mathbf{E}_n = \frac{1}{n}\left(\mathbf{U}^n - \mathbf{I}\right) \Rightarrow \begin{cases} \text{para}: n = 0 \Rightarrow \mathbf{E}_0 = \ln \mathbf{U} \quad , \text{Def. natural} \\ \text{para}: n = 1 \Rightarrow \mathbf{E}_1 = \mathbf{U} - \mathbf{I} \\ \text{para}: n = 2 \Rightarrow \mathbf{E} = \mathbf{E}_2 = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) \quad , \text{Def. de Green - St. Venant} \end{cases} \tag{2.9}$$

The *Eulerian strain* measured in the current configuration is expressed as the Almansi form. Then,

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right) \tag{2.10}$$

where $\mathbf{B}$ is the *Cauchy-Green left tensor* already defined, and $\mathbf{B}^{-1}$ is commonly called the *Finger tensor.* In case $|\mathbf{F} - \mathbf{I}| << 1$, all the strains previously defined coincide $\mathbf{E}_{ll} \cong \mathbf{e} \cong \boldsymbol{\varepsilon}$ and get closer to the *infinitesimal strain*,

$$\boldsymbol{\varepsilon} = \nabla^S \mathbf{u} = \frac{1}{2}\left(\nabla_0 \mathbf{u} + \nabla_0^T \mathbf{u}\right) \tag{2.11}$$

where $\mathbf{x} = \mathbf{x}_0 + \mathbf{u} \Rightarrow \mathbf{u} = \mathbf{x} - \mathbf{x}_0$ is satisfied, and $\mathbf{x}_0$ is the coordinates of the point $\mathbf{X}$ in the reference configuration and $\mathbf{u}$ is the relative displacement of such a point. Thus, the gradient is obtained as,

$$\nabla_0 \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}_0} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} - \mathbf{I}\right) = \left(\mathbf{F} - \mathbf{I}\right) = \mathbf{j} \tag{2.12}$$

From the latter and equation (2.11), the *infinitesimal strain and the strain gradient* are obtained as

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \frac{\partial}{\partial \mathbf{x}_0}\left(\mathbf{x}_0 + \mathbf{u}\right) = \left(\mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}_0}\right) = \mathbf{I} + \mathbf{j} \\ \boldsymbol{\varepsilon} &= \nabla^S \mathbf{u} = \frac{1}{2}\left(\mathbf{j} + \mathbf{j}^T\right) \end{aligned} \tag{2.13}$$

2.2.3 Relationships among mechanical variables

Given the transformation of the *strain gradient* **F,** which can relate the position of a point in a particular configuration to its image in any other configuration, an equivalence relationship can be established among all the other mechanical variables in one configuration with respect to their images corresponding to any other configuration. Therefore, the following tensor transformations[1,2,3,4] are defined

[4] Marsden J. And Hughes T. (1983). *Mathematical foundations of elasticity*. Prentice Hall, Enlewood Cliffs.

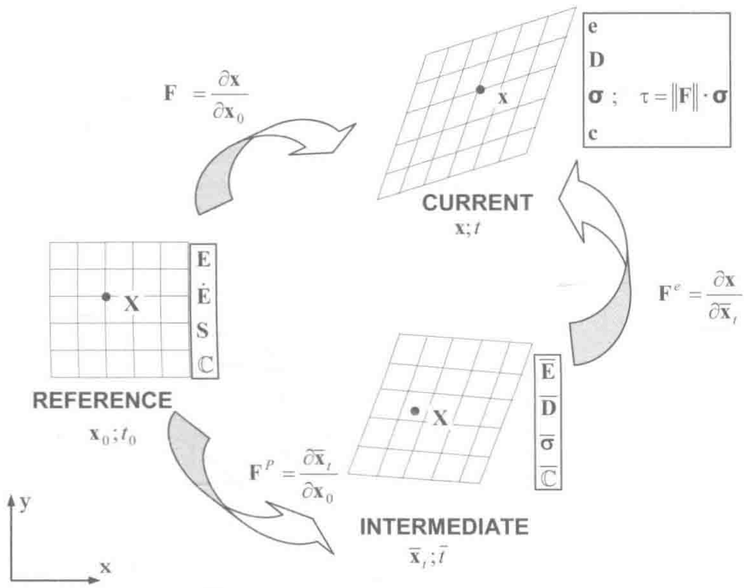

Figure 2.2 – Relationships among the mechanical variables in different configurations.

$$
\text{Tranformations}
\begin{cases}
\text{Covariant}
\begin{cases}
\text{push forward} & :\underset{\rightarrow}{\varphi}(\mathbf{A}_\#) = \mathbf{F}^{-T} \cdot (\mathbf{A}_\#) \cdot \mathbf{F}^{-1} \\
\text{pull back} & :\underset{\leftarrow}{\varphi}(\mathbf{A}_\#) = \mathbf{F}^{T} \cdot (\mathbf{A}_\#) \cdot \mathbf{F}
\end{cases} \\
\text{Contravariante}
\begin{cases}
\text{push forward} & :\vec{\varphi}(\mathbf{A}^\#) = \mathbf{F}^{T} \cdot (\mathbf{A}^\#) \cdot \mathbf{F} \\
\text{pull back} & :\overleftarrow{\varphi}(\mathbf{A}^\#) = \mathbf{F}^{-1} \cdot (\mathbf{A}^\#) \cdot \mathbf{F}^{-T}
\end{cases}
\end{cases}
\tag{2.14}
$$

where the operators and their expression as a function of the *gradient of strain* are shown in Figure 2.2 and $\mathbf{A}_\#$ and $\mathbf{A}^\#$ are *co-variant* generic tensors of second-order (deformation tensor $\mathbf{E} \leftrightarrow \mathbf{e}$) and *contravariant* (stress tensor $\mathbf{S} \leftrightarrow \boldsymbol{\tau}(\boldsymbol{\sigma})$) respectively. Particularly, the following transformations are obtained for the transportation of the stress-deformation and constitutive tensors[1],

$\mathbf{e} = \underset{\rightarrow}{\phi}(\mathbf{E})$	$e_{ij} = F_{iI}^{-T} E_{IJ} F_{jJ}^{-1}$	$\mathbf{e} = \mathbf{F}^{-\mathbf{T}} \cdot \mathbf{E} \cdot \mathbf{F}^{-1}$
$\mathbf{E} = \underset{\leftarrow}{\phi}(\mathbf{e})$	$E_{IJ} = F_{iI}^{T} e_{ij} F_{jJ}$	$\mathbf{E} = \mathbf{F}^{\mathbf{T}} \cdot \mathbf{e} \cdot \mathbf{F}$
$\boldsymbol{\tau} = \vec{\phi}(\mathbf{S})$	$\tau_{ij} = F_{iI} S_{IJ} F_{jJ}^{T}$	$\boldsymbol{\tau} = \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{T}$
$\mathbf{S} = \overleftarrow{\phi}(\boldsymbol{\tau})$	$S_{IJ} = F_{iI}^{-1} \tau_{ij} F_{jJ}^{-T}$	$\mathbf{S} = \mathbf{F}^{-1} \cdot \tau \cdot \mathbf{F}^{-\mathbf{T}}$
$\mathrm{c} = \vec{\phi}(ℂ)$	$c_{ijkl} = F_{iI} F_{jJ} F_{kK} F_{lL} C_{IJKL}$	
$ℂ = \overleftarrow{\phi}(\mathrm{c})$	$C_{IJKL} = F_{iI}^{-1} F_{jJ}^{-1} F_{kK}^{-1} F_{lL}^{-1} c_{ijkl}$	

Table 2.1 Kinematics: relation among tensors of the current and reference configurations.