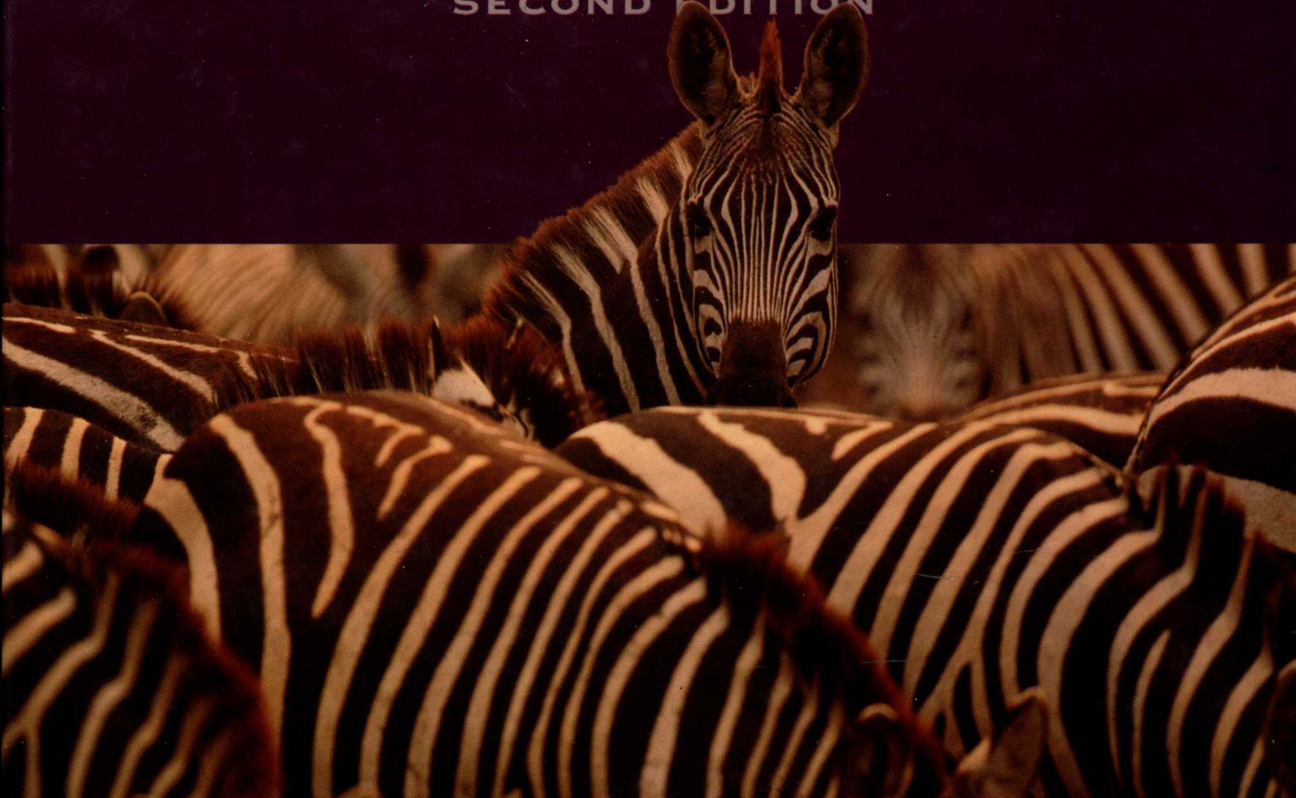
MATHEMATICS
FOR ELEMENTARY TEACHERS
SECOND EDITION



SYBILLA BECKMANN

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FOR ELEMENTARY TEACHERS

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UNIVERSITY OF GEORGIA

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To Will, Joey, and Arianna

Foreword by Roger Howe, Ph.D., Yale University

We owe a debt of gratitude to Sybilla Beckmann for this book.

Mathematics educators commonly hear that teachers need a "deep understanding" of the mathematics they teach. In this text, this pronouncement is not mere piety, it is the guiding spirit.

With the 1989 publication of its *Curriculum and Evaluation Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) initiated a new era of ferment and debate about mathematics education. The NCTM *Standards* achieved widespread acceptance in the mathematics education community. Many states created or rewrote their standards for mathematics education to conform to the NCTM *Standards*, and the National Science Foundation funded large-scale curriculum development projects to create mathematics programs consistent with the *Standards*' vision.

But this rush of activity largely ignored a major lesson from the 1960s' "New Math" era of mathematics education reform: in order to enable curricular reform, it is vital to raise the level of teachers' capabilities in the classroom. In 1999, the publication of *Knowing and Teaching Elementary Mathematics* by Liping Ma finally focused attention on teachers' understanding of mathematics principles they were teaching. Ma adapted interview questions (originally developed by Deborah Ball) to compare the basic mathematics understanding of American teachers and Chinese teachers. The differences were dramatic. Where American teachers' understanding was foggy, the Chinese teachers' comprehension was crystal clear. This vivid evidence showed that the difference in Asian and American students' achievement, revealed in many international comparisons, correlated to a difference in the mathematical knowledge of the teaching corps.

The Mathematical Education of Teachers, published by the Conference Board on the Mathematical Sciences (CBMS), was one response to Ma's work. Its first recommendation gave official voice to the dictum: "Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach." This report provided welcome focus on the problem, but the daunting task of creating courses to fulfill this recommendation remained.

Sybilla Beckmann has risen admirably to that challenge. Keeping mathematical principles firmly in mind while listening attentively to her students and addressing the needs of the classroom, she has written a text that links mathematical principles to their day-to-day uses. For example, in Chapter 5, Multiplication, the first section is devoted to the meaning of multiplication. First, it is defined through grouping: $A \times B$ means "the number of objects in A groups of B objects each." Beckmann then analyzes other common situations where multiplication arises to show that the definition applies to each. The section problems, then, do not simply provide practice in multiplication; they require students to show how the definition applies.

Subsequent sections continue to connect applications of multiplication (e.g., finding areas, finding volumes) to the definition. This both extends students' understanding of the definition and unites varied applications under a common roof. Reciprocally, the applications are used to develop the key properties of multiplication, strengthening the link between principle and practice. In the next chapter, the definition of multiplication is revisited and adapted to include fraction multiplication as well as whole numbers. Rather than emphasizing the procedure for multiplying fractions, this text focuses on how the procedure follows from the definitions.

Here and throughout the book, students are taught not merely specific mathematics, but the coherence of mathematics and the need for careful definitions as a basis for reasoning. By inculcating the point of view that mathematics makes sense and is based on precise language and careful reasoning, this book conveys far more than knowledge of specific mathematical topics: it can transmit some of the spirit of doing mathematics and create teachers who can share that spirit with their students. I hope the book will be used widely, with that goal in mind.

Preface

INTRODUCTION

I wrote this book to help future elementary school teachers develop a deep understanding of the mathematics that they will teach. It is easy to think that elementary school mathematics is simple and that it shouldn't require college-level study in order to teach it well. But to teach mathematics well, teachers must know more than just *how* to carry out basic mathematical procedures; they must be able to explain *why* mathematics works the way it does. Knowing why requires a much deeper understanding than knowing how. For example, it is easy to multiply fractions—multiply the tops and multiply the bottoms—but why do we multiply fractions that way? After all, when we *add* fractions, we can't just add the tops and add the bottoms. The reasons we can't in the one case and can in the other are not obvious; they require study. By learning to explain why mathematics works the way it does, teachers will learn to make sense of mathematics. I hope they will carry this "sense of making sense" into their own future classrooms.

Since I believe in deep understanding, the book focuses on "explaining why." Prospective elementary school teachers will learn to explain why the standard procedures and formulas of elementary mathematics are valid, why nonstandard methods can also be valid, and why other seemingly plausible ways of reasoning are not correct. The book emphasizes key concepts and principles, and it guides prospective teachers in giving explanations that draw on these key concepts and principles. In this way, teachers will come to organize their knowledge around the key concepts and principles of mathematics so that they will be able to help their students do likewise.

A number of activities and problems examine common misconceptions. Since most misconceptions have a certain plausibility about them, it is important to understand what makes them mathematically incorrect. By examining what makes misconceptions incorrect, teachers deepen their understanding of key concepts and principles, and they develop their sense of valid mathematical reasoning. I also hope that, by studying and analyzing these misconceptions, teachers will be able to explain to their students why an erroneous method is wrong, instead of just saying "You can't do it that way."

A number of activities and problems examine calculation methods that are nonstandard, but nevertheless correct. When explaining why nonstandard methods are correct, teachers have further opportunities to draw on key concepts and principles and to see how these concepts and principles underlie calculation methods. By examining nonstandard methods, teachers also learn that there can be more than one correct way to solve a problem and that it is neither convention nor authority, but rather valid logical reasoning, that determines whether a method is correct. I hope that by having studied and analyzed a variety of valid solution methods, teachers will be prepared to value their students' creative mathematical activity. It is surely discouraging to a student who has found an unusual but correct solution method to be told the method is incorrect. It also conveys to the student entirely the wrong message about what mathematics is.

Studying nonstandard methods of calculation provides valuable opportunities, but the common methods still deserve to be studied and appreciated. Why? These methods are remarkably clever and make highly efficient use of underlying principles. Because of these methods, we know that a wide range of problems can always be solved straightforwardly. The common methods are major human achievements and part of the world's heritage; like all mathematics, they are especially wonderful because they cross boundaries of culture and language.

In addition to knowing how to explain mathematics, prospective teachers should also know how mathematics is used. Therefore, this book emphasizes knowing which kinds of problems can be solved by the four basic operations of addition, subtraction, multiplication, and division. A common complaint of practicing teachers is that their students simply guess which operations to use in order to solve a problem. As a result, teachers must know and be able to convey what the four operations mean, and they must be able to write a variety of story problems for the operations. It is also helpful for teachers to know how mathematics is applied in science and economics. To this end, I have included various applications, such as using proportions to compare prices across years with the Consumer Price Index, using visualization skills to explain the phases of the moon, using spheres to explain how the Global Positioning System works, and using random samples to estimate populations with the capture–recapture method.

With this unique approach, I believe that this book is an excellent fit for the recommendations of the Conference Board of the Mathematical Sciences regarding the mathematical preparation of teachers. I also believe that the book helps prepare teachers to teach in accordance with the principles and standards of the National Council of Teachers of Mathematics.

PEDAGOGICAL AND CONTENT FEATURES

Class Activities were developed alongside the text and are central and integral to the text. All good teachers of mathematics know that mathematics is not a spectator sport—we can't learn mathematical ideas solely by watching someone else present them. Instead, we must actively think through mathematical ideas in order to make sense of them for ourselves. When students work on problems within the class activities, first on their own, then in a pair or a small group, and then within a whole class discussion, they have a chance to think through the mathematical ideas several times. By discussing mathematical ideas and explaining their solution methods to each other, students can deepen and extend their thinking. As every mathematics teacher knows, you really learn mathematics when you have to explain it to someone else. The activities are presented in a separate activities manual correlated with the main textbook.

To get a feel for the class activities, take a look at the following:

- Class Activity 3Q: Can We Reason This Way? (page 52)
- Class Activity 4K: A Third Grader's Method of Subtraction (page 82)
- Class Activity 5X: Showing the Algebra in Mental Math (page 153)
- Class Activity 6H: Explaining Why We Put the Decimal Point Where We Do When We Multiply Decimals (page 172)
- Class Activity 7J: Interpreting Standard Long Division from the "How Many in Each Group?" Viewpoint (page 202)
- Class Activity 8P: How Big Is the Reflection of Your Face in a Mirror? (page 271)
- Class Activity 11H: Explaining Why the Area Formula for Triangles Is Valid (page 380)
- Class Activity 12H: Flower Designs (page 437)

- Class Activity 13M: Solving Story Problems with Strip Diagrams and with Equations (page 487)
- Class Activity 14C: Using Random Samples to Estimate Population Size by Marking (Capture–Recapture) (pages 542–544)

Practice Problems give students the opportunity to try out problems. Solutions appear in the text immediately after the Practice Problems. These solutions provide students with numerous examples of the kinds of good explanations they should learn to write. By attempting the Practice Problems themselves and then checking their solutions against the solutions provided, students will be better prepared to provide good explanations in their homework.

Problems are opportunities for students to explain the mathematics they have learned, without being given an answer at the end of the text. Problems are typically assigned as homework. Solutions appear in the Instructor's Solutions Manual and can be provided online for students at the discretion of the instructor.

To get a feel for the problems, take a look at the following:

- Problem 5 on page 57 of Section 2.4 (Comparing Decimal Numbers)
- Problem 4 on page 75 of Section 3.1 (The Meaning of Fractions)
- Problem 13 on page 149 of Section 4.2 (Why the Common Algorithms for Adding and Subtracting Decimal Numbers Work)
- Problem 4 on page 246 of Section 5.6 (Mental Math, Properties of Arithmetic, and Algebra)
- Problem 4 on page 270 of Section 6.1 (Multiplying Fractions)
- Problem 16 on page 374 of Section 7.6 (Ratio and Proportion)
- Problem 10 on page 540 of Section 10.3 (Calculating Perimeters)
- Problem 4 on page 644 of Section 11.11 (Areas, Volumes, and Scaling)
- Problem 22 on page 728 of Section 13.2 (Solving Equations Using Number Sense, Strip Diagrams, and Algebra)

NEW! A **chapter summary** and study items are provided at the end of each chapter to help students organize their thinking and focus on key ideas as they study for exams.

NEW! The **"core" icon** denotes central material. These problems and activities (or similar ones) are recommended for mastery of the material.

NEW! * This **"extension" icon** denotes material that is beneficial to cover but not critical. These problems and activities are either more challenging, involve an extended investigation, or are designed to extend students' thinking beyond the central areas of study.

Students will use **visualization** not only in traditional mathematical contexts but also in order to understand basic astronomical phenomena, such as the phases of the moon, the reason for the seasons, and the rotation of the earth around its axis every day.

There is ample coverage of **volume and surface area**. Both the text and the activities manual promote a deeper understanding of these concepts with the aid of hands-on exploration.

Unique content in Chapter 13 on functions and algebra introduces U.S. teachers to the impressive diagrammatic method presented in the math texts for grades 3–6 used in Singapore, whose children get the top math scores in the world. This method helps students make sense of and solve a variety of **algebra and other word problems** without using variables. The text helps students see the relationship between the Singaporean diagrammatic method and standard algebraic problem-solving methods.

The book is **organized around the operations** instead of around the different types of numbers. In my view, there are two key advantages to focusing on the operations. The first advantage is a more advanced, unified perspective, which emphasizes that a given operation (addition, subtraction, multiplication, or division) retains its meaning across all the different types of numbers. Prospective teachers who have already studied numbers and operations in the traditional way for years will find that this method enables them to take a broader view and to consider a different perspective. A second advantage is that fractions, decimals, and percents—traditional weak spots—can be studied repeatedly throughout a course, rather than only at the end. The repeated coverage of fractions, decimals, and percents allows students to become gradually used to reasoning with these numbers, so they aren't overwhelmed when they get to multiplication and division of fractions and decimals.

The **problem-solving chapter** (Chapter 1) provides a brief introduction to the process of problem solving and to writing good explanations. As students progress through the chapters, they learn new problem-solving methods and become more skilled at solving problems and at explaining their solution methods. Solving problems and explaining why methods of solution are valid are central to every chapter.

Arithmetic in bases other than 10 is not explicitly discussed, but other bases are implied in several activities and problems. For example, see Class Activity 4I: Regrouping with Dozens and Dozens of Dozens, which involves base 12, and Class Activity 4J: Regrouping with Seconds, Minutes, and Hours, which involves base 60. Problems 7–11 on pages 148 and 149 of Section 4.2 also involve the idea of other bases. These activities and problems allow students to grapple with the significance of the base in place value without getting bogged down in the mechanics of arithmetic in other bases.

The **counting methods of probability** are uniquely explained. When determining probabilities, we must frequently multiply to determine all possible outcomes of an experiment. Rather than have students simply accept a rule stating that we should multiply to determine a total number, we encourage students to understand where this rule comes from: It follows directly from the meaning of multiplication. Therefore, methods of counting that rely on multiplication are in Section 5.1, The Meaning of Multiplication and Ways to Show Multiplication.

CONTENT CHANGES FOR THE SECOND EDITION

This text has been extensively enhanced, first through the use of new representations. Strip diagrams, percent diagrams, area pictures, and additional tables and number lines will help students solve word problems and understand percentages and ratios.

In revising this text, I have also responded to educational studies on decimals, percentages, story problems, and ten-structured pictures. Chapter 2 includes new material on representing decimals as lengths, as well as descriptions of common misconceptions children have in comparing decimal numbers and a new activity to help prospective teachers understand those misconceptions. Chapter 4 offers new material about the types of addition and subtraction story problems and an improved discussion on addition and subtraction with negative numbers and with number lines. Section 7.6 now discusses ratio from two points of view about division, including several types of representations to increase understanding. And Chapter 15 now uses area pictures to support probability calculations.

I am enthusiastic about these additions and am sure students and professors will be as well.

SUPPLEMENTS

Activities Manual

0-321-44976-2

This activities manual includes activities designed to be done in class or outside of class. These activities promote critical thinking and discussion and give students a depth of understanding and perspective on the concepts presented in the text.

Instructor's Resource and Testing Manual

0-321-44861-8

This manual includes teaching tips in each chapter, tips and solutions for the activities, sample test problems, and course organization and grading suggestions.

Instructor's Solutions Manual

0-321-45565-7

The Instructor's Solutions Manual contains worked-out solutions to all problems in the text.

Addison-Wesley Math Tutor Center

0-201-72170-8

The Addison-Wesley Math Tutor Center provides free tutoring through a registration number that can be packaged with a new textbook or purchased separately. The center is staffed by qualified college mathematics instructors and is accessible via toll-free telephone, toll-free fax, e-mail, and the Internet. For more information, visit *www.aw-bc.com/tutorcenter*.

Acknowledgments

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Sybilla Beckmann
Athens, Georgia

Brief Contents

Contents

Throughout the text, each section concludes with Practice Problems, Answers to Practice Problems, and Problems.