New Advances in Research on H-matrices

（H-矩阵研究的新进展）

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Preface

The class of H-matrices is an important class of matrices with special structure, not only including diagonally dominant matrices, generalized diagonally dominant matrices and M-matrices, but also intersecting with positive stable matrices, P-matrices and positive definite matrices. In 1937 the concepts of nonsingular M-matrix and strong H-matrix were introduced firstly by Ostrowski for the research of the convergence of iterative methods for linear and nonlinear systems and spectral theory. Later, Fiedler and Ptak generalized these concepts to possible singular M-matrices and weak H-matrices. Recently, the definition of H-matrices has been extended to encompass a wider set, known as the set of general H-matrices. Such a class of matrices includes the class of strong H-matrices (nonsingular H-matrices defined by Ostrowski) and the class of weak H-matrices (singular H-matrices defined by Fiedler and Ptak) which contains the class of mixed H-matrices and the class of degenerate H-matrices. On the other hand, the concept of H-matrix is generalized to the block matrix form in two directions since beautiful properties and wide applications of H-matrices. One is block H-matrices based on some matrix norm; another is generalized H-matrices defined by using the Loewner partial order. The two generalizations of H-matrices has an important theoretical and numerical applications in large matrix computations, and thus, enriches the class of H-matrices.

The past two decades have witnessed some new advances of theoretical and computational topics on H-matrices. With the development of computer technology and advances in the research on special matrices, the class of H-matrices has been not only an indispensable and versatile tool but also applied to a wide range of computational mathematics, applied mathematics, mathematical physics, finance and economics, optimal control, biological systems, neutral network, computer network and engineering design, etc.

This book aims at providing the reader with a series of advances on a wide variety of general H-matrices, composed of more than ten papers published by the author in the last decade, and contributing to a summary report of the author on the research in numerical linear algebra. The goal is to introduce some new advances in the research on the theoretical and numerical properties of H-matrices to the reader in the field of applied mathematics, computational mathematics and so on, provide insights into how to analyze and solve some problems about special matrices that arise in a variety of practical applications. The topics included are nonsingularity/singularity, the Schur complement, the eigenvalue distribution, the convergence theory of classic iterative methods on nonstrictly diagonally dominant matrices and general H-matrices, and the related theory of block diagonally dominant matrices, block H-matrices, generalized H-matrices and extended H-matrices as well.

There are excellent works that introduce nonnegative matrices, nonsingular M-matrices, diagonally dominant matrices and strong H-matrices, providing the reader with all the basics on matrix analysis, matrix computations, and a number of insightful engineering applications. This book is targeted at advanced graduate students, or advanced researchers that are already familiar with the basics of linear algebra. It can be used as a textbook for an advanced graduate course emphasizing special matrices, or as a complement to an introductory textbook that provides systematically theoretical and numerical properties of H-matrices. It can also be used for self-study to become acquainted with the state of-the-art in a wide variety of topics on linear algebra.

This book contains 2 diverse parts. The first part is to discuss 6 topics of point H-matrices and thus is composed of 6 chapters covering a large variety of H-matrices. After introducing the development, applications and some notations and related preliminary results of diagonally dominant matrices and H-matrices in Chapter 1, we proposed some new theoretical results on nonsingularity/singularity in Chapter 2, the Schur complements and the generalized Schur complements in Chapter 3 of nonstrictly diagonally dominant matrices and general H-matrices. In Chapter 4, we investigate the eigenvalue distribution on nonstrictly diagonally dominant matrices, general H-matrices and their (generalized) Schur complements. Meanwhile, we propose some interesting and significant results on the generalized eigenvalue distribution of diagonally dominant matrix pair and H-matrix pair. In Chapter 5, some convergence theorems on several basic iterative methods including Jacobi iterative method, Gauss- Seidel iterative methods, SOR iterative methods and AOR iterative method are established for the linear systems whose coefficient matrices are nonstrictly diagonally dominant matrices and general H-matrices, especially, weak H-matrices. The class of radial matrices that related to diagonally equipotent matrices is investigated in Chapter 6 to derive some theoretical and applied results.

The second part contains 3 chapters to discuss generalizations of H-matrices, and mainly derive two class of H-matrices with the block matrix form: one is block diagonally dominant matrices and block H-matrices; the other is generalized H-matrices and extended H-matrices. A brief account of each chapter is given next. Chapter 7 introduces the background and development of generalization of H-matrices, puts emphasis on the two generalizations of H-matrices and then presents some related notations and preliminary results on block diagonally dominant matrices, block H-matrices, generalized H-matrices and extended H-matrices as well. Some theoretical and numerical properties including nonsingularity/singularity, the Schur complements and the eigenvalue distribution are established in Chapter 8 for block diagonally dominant matrices and block H-matrices. In Chapter 9, some convergence results on some block iterative methods including block Jacobi iterative method, block Gauss-Seidel iterative method, block SOR iterative method, block AOR iterative method and parallel multisplitting block iterative methods are established for generalized H-matrices and extended H-matrices, then some applications to special cases are shown from the computations of partial differential equations and some numerical examples demonstrate the effectiveness of these results.

Due to the relationship of the book chapters, it is possible to organize the book into thematic areas and each chapter can be treated dependently of the others.

In order to satisfy readers with this special interest, we list, at the end of the book, as many up-to-date references as possible.

At this time I would like to acknowledge some people who have helped me with various aspects of this book. Firstly, I thank my supervisors, Professors Chengxian Xu and Yaotang Li, for their guidance during my academic years at School of Science, Xi'an Jiaotong University and School of Mathematics and Physics, Yunnan University, respectively. Meanwhile, I would like to thank my co-supervisors, professors Zongben Xu and Michele Benzi, for their selfless help and valuable suggestions during my post-doctoral years at Institute of Information and System Science, Xi'an Jiaotong University and my visiting period at the Department of Mathematics and Computer Science, Emory University, USA, respectively. During my research years in numerical linear algebra I benefited from Professors Jianzhou Liu, Zhongzhi Bai, Yongzhong Song, Kolotilina, Elsner, Nabben, Fuzhen Zhang, Jiaju Zhang and so on.

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I specially thank my wife, associate professor Shuanghua Luo and my daughter Jiaxuan Zhang, for their support. It is not hard to imagine how much they have sacrificed in family life during the past years. I would like to dedicate this book to my wife and daughter.

Though I had made a good effort to write this book, it is surely that there are still some errors. Although I do so sadly, I take the blame for all of these mistakes. I would appreciate it if you would send any mistakes that you find to cyzhang08@126.com. Thank you.

Chengyi Zhang
May 12, 2017

Contents

PART TWO GENERALIZATIONS OF H-MATRICES

PART ONE

POINT H-MATRICES

Chapter 1
Introduction

With the rapid development of modern science and technology and the continuous improvement in computer performance, numerical algebra plays an increasingly important role in solving many of the major practical problems that arise in engineering, technology and production. Moreover, the solution of many important problems comes down to matrices or linear systems related to matrices.

The term "matrix" is introduced firstly by Sylvester (1850), with Latin as "womb", as the name of the array of numbers. Matrix theory is not only an important tool of mathematics and many disciplines of science and engineering, but also has rich creativity in its own research. The birth and development of many new theories, methods and techniques are the result of creative application and popularization of matrix theory. Matrix theory is an important branch of mathematics, and has become a powerful tool for modern science and technology to deal with a large number of finite dimensional space form and quantity relationship. For example, system engineering, optimization methods and stability theory, are closely related with the matrix.

Special matrices, which have a special property or special structure of the matrix, are occupying a very important position in the field of mathematics and related disciplines. Whether in the field of theory or in the application field special matrices have a unique charm. As one of main research topics of numerical algebra, special matrices have been applied to a wide range of computational mathematics, applied mathematics, mathematical physics, finance and economics, optimal control, biological systems and computer network, etc.

The class of H-matrices, proposed by American mathematician Ostrowski (1937) to study the convergence of matrix iterative schemes, is a natural extension of the class of M-matrices (see Berman and Plemmons (1979)) and one of the most important classes of matrices in special matrices. This class of matrices, including diagonally dominant matrices, generalized diagonally dominant matrices and M-matrices, furthermore, intersecting with positive stable matrices, P-matrices and positive definite matrices, has many significant and wide applications, e.g.

(1) In numerical linear algebra, H-matrices are mainly applied to the establishment of convergence criteria for iterative methods for the solution of large sparse systems of linear equations arising naturally in some discretizations of partial diffe-

rential equations, and as such are well-studied in scientific computing, see Berman and Plemmons (1979); Saad (1996); Demmel (1997); Varga (2000).

(2) M-matrices, belonging to H-matrices, widely arise in the input-output analysis model in economics, the linear complementary problem in optimization, Markov chain in probability and statistics, and so on, see Berman and Plemmons (1979). For example, in the analysis of Leontief's input-output model, when the coefficient matrix $A = I - T$ of the input-output equation of equilibrium $Ax = (I - T)x = b$ is a nonsingular M-matrix, the model is very lucrative.

(3) H-matrices not only are applied widely to numerical linear algebra, control theory, optimization theory and methods, numerical solutions of differential equation, economic model, probability and statistics as well, but also have an important role in stability analysis of neural network, large ecosystem, dynamic system and so on, see Liao (2000); Wang et al. (2004); Liao et al. (2007); Zhang (2010).

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In what follows H-matrices will be introduce by two aspects including diagonally dominant matrices and H-matrices. Finally, research status and classification of H-matrices will be taken into account.

1.1 Speaking from diagonally dominant matrices

In order to study nonsingularity of a matrix, strictly diagonal dominance theorem was first published by Lévy (1881)(see Schneider (1977)) under the assumption on the matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with $a_{ii} < 0$ and $a_{ij} \geqslant 0$ for $i \neq j$, $i, j = 1, 2, \cdots, n$. In a subsequent paper, Desplanques (1886) gave a proof of the general case of this theorem. Later, the strictly diagonal dominance is restated in Nekrasov (1892) and Mebmke (1892). The strictly diagonally dominant matrix and related theorem were rediscovered independently by Hadamard (1903) and Minkowski (1900, 1907).

In the next a few decades a series of concepts, such as diagonally dominant matrix with SC property, irreducibly diagonally dominant matrix, diagonally dominant matrix with nonzero-entry chain, and semi-strictly diagonally dominant matrix, α-diagonally dominant matrix and doubly diagonally dominant matrix have been proposed for the research of nonsingularity of diagonally dominant matrices (see Taussky (1949); Shivakumar (1974); Berman and Plemmons (1979); You (1981); Horn and Johnson (1985)). Almost the same time, generalized diagonally dominant matrices including generalized strictly diagonally dominant matrices had been introduced and many beautiful properties had been derived.

Zhang et al. (2007, 2008) proposed concepts of diagonally equipotent matrices and generalized diagonally equipotent matrices to derive some necessary and sufficient conditions on singularity of diagonally dominant matrices and generalized diagonally dominant matrices.

In the following the concepts and classification of diagonally dominant matrices will be introduced for convenience of subsequent description.

$\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) will be used to denote the set of all $m\times n$ complex (real) matrices. $\mathbb{Z}$ denotes the set of all integers. Let $\langle n\rangle = \{1,2,\cdots,n\}\subset\mathbb{Z}$.

For $n\geqslant 2$, an $n\times n$ complex matrix A is reducible if there exists an $n\times n$ permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{1.1}$$

where A_{11} is an $r\times r$ submatrix and A_{22} is an $(n-r)\times(n-r)$ submatrix, where $1\leqslant r<n$. If no such permutation matrix exists, then A is called irreducible. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Definition 1.1 A matrix $A\in\mathbb{C}^{n\times n}$ is called diagonally dominant by row if

$$|a_{ii}| \geqslant \sum_{j=1,j\neq i}^{n} |a_{ij}| \tag{1.2}$$

holds for all $i\in\langle n\rangle$. If inequality in (1.2) holds strictly for all $i\in\langle n\rangle$, A is called strictly diagonally dominant by row. If A is irreducible and the inequality in (1.2) holds strictly for at least one $i\in\langle n\rangle$, A is called irreducibly diagonally dominant by row. If (1.2) holds with equality for all $i\in\langle n\rangle$, A is called diagonally equipotent by row.

$D_n(\mathrm{SD}_n,\ \mathrm{ID}_n)$ and DE_n will be used to denote the sets of all $n\times n$ (strictly, irreducibly) diagonally dominant matrices and the set of all $n\times n$ diagonally equipotent matrices, respectively.

Definition 1.2 A matrix $A\in\mathbb{C}^{n\times n}$ is called lower semi-stricly diagonally dominant by row if A is diagonally dominant by row, i.e., (1.2) holds for all $i\in\langle n\rangle$, and if, moreover

$$|a_{ii}| \geqslant \sum_{j=1}^{i-1} |a_{ij}| \tag{1.3}$$

holds for all $i\in\langle n\rangle$. It is called semi-strictly diagonally dominant if there exists a permutation matrix Q such that $\tilde{A}=QAQ^{\mathrm{T}}$ is lower semi-strictly diagonally dominant.

LSSD_n and SSD_n will be used to denote the sets of all $n\times n$ lower semi-strictly diagonally dominant matrices and the set of all $n\times n$ semi-strictly diagonally dominant matrices, respectively.

Definition 1.3 A matrix $A \in \mathbb{C}^{n\times n}$ is called generalized diagonally dominant (GDD) if there exist positive constants α_i, $i \in \langle n\rangle$, such that

$$\alpha_i|a_{ii}| \geqslant \sum_{j=1,j\neq i}^{n} \alpha_j|a_{ij}| \tag{1.4}$$

holds for all $i \in \langle n\rangle$. If inequality in (1.4) holds strictly for all $i \in \langle n\rangle$, A is called generalized strictly diagonally dominant. If (1.4) holds with equality for all $i \in \langle n\rangle$, A is called generalized diagonally equipotent.

Lemma 1.4(see Berman and Plemmons (1979); Zhang (1980)) *A matrix $A \in \mathbb{C}^{n\times n}$ is generalized (strictly) diagonally dominant if and only if there exists a positive diagonal matrix D such that $D^{-1}AD$ is (strictly) diagonally dominant.*

We denote the sets of all $n\times n$ generalized (strictly) diagonally dominant matrices and the set of all $n\times n$ generalized diagonally equipotent matrices by $\mathrm{GD}_n(\mathrm{GSD}_n)$ and GDE_n, respectively.

Definition 1.5 A matrix A is called nonstrictly diagonally dominant (NDD), if either (1.2) or (1.4) holds with equality for at least one $i \in \langle n\rangle$.

Remark 1 Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be nonstrictly diagonally dominant and $\alpha = \langle n\rangle - \alpha' \subset \langle n\rangle$. If $A(\alpha)$ is a (generalized) diagonally equipotent principal submatrix of A, then the following hold:

(1) $A(\alpha,\alpha') = 0$, which shows that A is reducible;

(2) $A(i_1) = (a_{i_1i_1})$ being (generalized) diagonally equipotent implies $a_{i_1i_1} = 0$.

The remark above gives the following lemma.

Lemma 1.6 *If a matrix $A \in D_n(\mathrm{GD}_n)$ has a (gneralized) diagonally equipotent principal submatrix $A(\alpha)$ for $\alpha \subset \langle n\rangle$, then A is reducible.*

Remark 2 Definition 1.2 and Definition 1.3 show that

$$D_n \subset \mathrm{GD}_n \quad \text{and} \quad \mathrm{GSD}_n \subset \mathrm{GD}_n.$$

1.2 H-matrices

In the research of the convergence of iterative methods for linear and nonlinear systems and spectral theory, Ostrowski (1937) firstly introduced the concept of nonsingular M-matrix and nonsingular H-matrix. Since their beautiful properties and wide applications, the research on nonsingular M-matrix and nonsingular H-matrix attract considerable attention, resulting in numerous papers devoted to various theoretical extensions of this significant topic, see Fiedler and Ptak (1962, 1967); Robert (1969); Varga (1976); Plemmons (1977); Berinan and Plemmons (1979); Johnson (1982); Polman (1987); Song (1994); Alanelli and Hadjidimos (2006), etc. Later, Fiedler and

Ptak (1962, 1967) extended this concept to possible singular M-matrices and singular H-matrices. As a result, the theory of H-matrices has been enriched.

Recently, the definition for H-matrices has been extended to encompass a wider set, known as the set of general H-matrices. In some recent papers, see Bru et al. (2008, 2009, 2012), a partition of the $n \times n$ general H-matrix set, $\mathscr{H}_n$, into three mutually exclusive classes was obtained: the invertible class, $\mathscr{H}_n^I$, where all general H-matrices are nonsingular, the singular class, $\mathscr{H}_n^{\rm S}$, formed only by singular H-matrices, and the mixed class, $\mathscr{H}_n^{\rm M}$, in which singular and nonsingular H-matrices coexist.

Here, we give a slightly different partition: the general H-matrix set H_n are partitioned into two mutually exclusive classes: the strong class, $H_n^{\rm S}$, where the comparison matrices of all general H-matrices in $H_n^{\rm S}$ are nonsingular M-matrices, and the weak class, $H_n^{\rm W}$, where the comparison matrices of all general H-matrices in $H_n^{\rm W}$ are singular M-matrices. A further partition of the $n \times n$ weak H-matrix set, $H_n^{\rm W}$, into two mutually exclusive classes was obtained: the degenerate class, $H_n^{\rm D}$, formed only by weak H-matrices with at least one zero diagonal entry, and the mixed class, $H_n^{\rm M}$, in which all diagonal entries of weak H-matrices are nonzero, what's more, singular and nonsingular H-matrices coexist.

Although the former partition avoid effectively the use of "nonsingular", "invertible" is the same as "nonsingular" in the sense of matrix but "invertible H-matrices" is not the same as "nonsingular H-matrices" in the former partition. This is easy to cause confusion of notations. Hence, we adopt the latter partition.

In what follows we introduce the notations of H-matrices and classification.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called nonnegative if $a_{ij} \geqslant 0$ for all $i, j \in \langle n \rangle$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a Z-matrix if $a_{ij} \leqslant 0$ for all $i \neq j$. We will use Z_n to denote the set of all $n \times n$ Z-matrices. A matrix $A = (a_{ij}) \in Z_n$ is called an M-matrix if A can be expressed in the form $A = sI - B$, where $B \geqslant 0$, and $s \geqslant \rho(B)$, the spectral radius of B. If $s > \rho(B)$, A is called a nonsingular M-matrix; if $s = \rho(B)$, A is called a singular M-matrix. M_n, $M_n^{\bullet}$ and M_n^0 will be used to denote the set of all $n \times n$ M-matrices, the set of all $n \times n$ nonsingular M-matrices and the set of all $n \times n$ singular M-matrices, respectively. It is easy to see that

$$M = M_n^{\bullet} \cup M_n^0 \quad \text{and} \quad M_n^{\bullet} \cap M_n^0 = \varnothing. \tag{1.5}$$

The comparison matrix of a given matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } \ i = j, \\ -|a_{ij}|, & \text{if } \ i \neq j. \end{cases}$$

It is clear that $\mu(A) \in Z_n$ for a matrix $A \in \mathbb{C}^{n\times n}$. The set of equimodular matrices associated with A, denoted by $\omega(A) = \{B \in \mathbb{C}^{n\times n} : \ \mu(B) = \mu(A)\}$. Note that both A and $\mu(A)$ are in $\omega(A)$. A matrix $A = a_{ij} \in \mathbb{C}^{n\times n}$ is called a general H-matrix if $\mu(A) \in M_n$ (see Bru et al. (2008, 2009, 2012)). If $\mu(A) \in M_n^{\bullet}$, A is called a strong