Mathematics
An Integrated View

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MATHEMATICS: AN INTEGRATED VIEW

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PREFACE

This book is an attempt to select some of the important ideas of mathematics, interweave them to make a unified pattern, yet leave some colorful strands protruding to encourage the reader to extend the pattern. Highly technical prerequisites would defeat these purposes. Though the text presupposes two or three years of high school mathematics, or the equivalent, it demands little specific knowledge or technique beyond elementary arithmetic. The need, instead, is for careful reading, active thinking, and the willingness to experiment with ideas and examples.

The choice and organization of the material is designed to provide an integrated introduction to or extension of some important aspects of modern mathematics with which an educated person should be familiar. Thus the book may prove helpful to able high school students, liberal arts students in junior and four-year colleges, elementary education majors and teachers, and others wishing to investigate mathematical ideas on their own.

The central core is a sequential treatment of sets, logic, and number systems. The book, however, stresses interrelations among these topics and also ties in many additional mathematical concepts. As a text, it contains enough material for a three-hour college course for nonmajors. It is tightly organized in two respects. First, the chapters and sections (except for the starred sections) should be studied in order, though the pace and the amount of emphasis on proving theorems may be varied. Second, the exercises form an integral part of the presentation, illustrating and extending the preceding concepts and questions and occasionally even introducing new ones. In this way the reader is able to become actively involved in understanding and applying the ideas presented, almost as if he or she were using a self-instruction program.

To allow for individual differences, most exercise sets contain a few starred, more challenging problems. In addition, five starred (optional) sections are included; they may be omitted or deferred without loss of continuity. Each chapter ends with a broad list of Suggestions for Investigation and a working bibliography of books and articles to encourage further exploration. The section immediately following the text contains selected answers to about half the unstarred problems, and is followed by a general bibliography and a reference list of all definitions and theorems by number and page.

Like most expository books, this one is the product of study, classroom experience, and personal contact with teachers, colleagues, students, and others to such an extent that a complete list of acknowledgements is impossible. Generous thanks are due to the students in my general mathematics classes, in particular to those who lugged around heavy preliminary editions, pointed out numerous errors and unclear passages, and made valuable suggestions for improvement. A mathematics major, Ms. Belinda Phillips, helped considerably in working out answers for many exercises. I am also grateful to the administration of Russell Sage College and to my colleagues for their encouragement in writing and using the materials; and to the typists and staff who produced them. Finally, I appreciate greatly the advice and technical assistance of the editors and staff of the Charles E. Merrill Publishing Company in converting the manuscript into a book.

Roland F. Smith

CONTENTS

1

The Nature of Mathematics

How old is mathematics? Is any mathematics really *new? What* is *mathematics, anyway? And what do mathematicians do besides adding numbers and drawing circles?*

Section 1.1 Mathematics: New or Old?

What comes immediately to mind when the word *mathematics* is mentioned? The response is likely to be in terms of personal experience; and it probably reveals more about the responder than about the subject of the inquiry. For some, mathematics connotes order, precision, logical reasoning, exciting challenges and discoveries, even beauty. But for many others it evokes the tasks of adding long columns of numbers, manipulating letters, solving equations, memorizing dull lists of definitions and proofs — at most mastering an unchanging body of more or less useful facts inherited from the Greeks and Arabians. In fact, the latter impression is not only quite uncomplimentary to mathematics, but it is also largely false: it is as much a travesty as to picture nursing as simply taking patients' temperatures or swimming as managing somehow to stay afloat in water.

The idea that mathematics is a cut-and-dried accretion of facts which one learns once and for all is epitomized in the phrase "as certain as two and two make four." (We shall see later that even this bit of arithmetic is not certain; in some circumstances one may claim that $2 + 2 = 10$, or $2 + 2 = 0$, or $2 + 2 = 1$.) It is true, of course, that the Greeks and the Arabians made signal contributions to mathematics that are relevant today. Indeed, evidences of written mathematics predating 3000 B.C. occur in Egypt, and a rudimentary sense of number probably goes back to primitive times.

In this connection, it is interesting to discuss the truth or falsity of the following two statements:

1. More *physics* has been discovered (or invented) since the death of Newton (1727) than in all previous recorded history.
2. More *mathematics* has been discovered (or invented) since the death of Newton (1727) than in all previous recorded history.

When college students who are not majoring in science confront these statements, they are likely to agree by a small majority that the first is true. As for the second statement, however, they are likely uncertain, and at best evenly divided when judging it true or false. Many of those who mark both statements true, moreover, admit that they are guessing or that they cannot justify their answers. Both statements *are* true, of course, and plenty of evidence is at hand to document them. As for mathematics, virtually any measure one chooses to apply shows that it is participating fully in the current "knowledge explosion."

One measure of the growth of mathematics is the volume of published materials, especially that concerned with new results. For example, the *Proceedings* of the American Mathematical Society, which principally contains newly created mathematics, appears monthly. The issue of April 1973 contained 222 pages with 44 articles; some issues have been considerably larger. This is *one* research publication of *one American* research organization: the Society publishes other research periodicals; several other American research societies publish journals; and many other journals are published in England, Germany, France, Russia, Japan, China, and elsewhere.

A related statistic is the number of papers presented at professional meetings of mathematicians. The American Mathematical Society is *one* of the *American* organizations which has sponsored regular conferences, the largest of which has been the annual meeting. The program of the 1973 meeting contained 733 short papers and 12 major addresses; in addition, 109 papers were listed "by title" and summarized in writing.[1] We underline the point that the above data concern only one meeting, and note that the January, February, April, and June, 1973, issues of the *Notices* of the American Mathematical Society gave announcements of a total of 50 regular and special conferences on mathematics scheduled in the United States during the calendar year 1973, in addition to 58 meetings in Canada, Mexico, and overseas.

[1] *Notices*, American Mathematical Society, vol. 20, no. 1, January, 1973.

If one wishes to judge by the number of *new* fields or areas of mathematics since Newton's day, it is interesting to examine a recently devised classification scheme. The classification list contains 63 major headings, some with 80 or more subheadings. Of the major headings, 42 (67%) refer to fields whose development has occurred almost entirely since Newton's time. Of the remaining 21 fields, virtually all include a large proportion of subheadings which belong to the post-Newtonian era. This scheme is used in connection with the computerized "Mathematical Offprint Service"[2] to help mathematicians keep abreast of research activities in their particular specialties. Through this service, a mathematician can obtain weekly lists of all published articles in his field and reprints of those which especially interest him.

In 1966, a well-known mathematician and historian, K. O. May, presented a paper on the growth of mathematical literature during the past hundred years.[3] His conclusion was that mathematical literature is growing at an annual rate of about $2\frac{1}{2}\%$. Put another way, the body of mathematical literature doubles about every 28 years. This is a *continuous* rate of growth, comparable with the rate at which money would grow in a bank which compounds interest moment by moment, rather than two or four times a year. Assuming that the volume of published articles in mathematics is a reliable index of the growth of mathematical knowledge, we must conclude that the statement concerning post-Newtonian mathematics is much too conservative. Indeed, it might justifiably be replaced by one such as "More mathematics has been created within the past generation than in all previously recorded history."

From a different point of view, we could ask how many mathematicians are active in the United States now, or how many are entering mathematical professions annually. A crude measure of the latter is the number of persons who earn degrees in mathematics each year. A few statistics are presented in Table 1.1.

TABLE 1.1

Earned Degrees in Mathematics and Statistics

	Bachelor's		*Master's*		*Doctor's*	
Year	*Number*	*Percent**	*Number*	*Percent**	*Number*	*Percent**
1959–60	11,399	9.9	1757	9.6	303	6.5
1964–65	19,547	14.3	4290	14.3	688	8.3
1969–70	28,986	15.7	7095	16.8	1343	9.7
1970–71**	29,940	15.9	7770	17.6	1480	10.1

SOURCE: *Projections of Educational Statistics to 1980–81*, 1971 ed., Office of Education, U. S. Department of Health, Education and Welfare (Washington, D.C.: U. S. Government Printing Office, 1972), excerpts from pp. 47, 50, 53.

*The base is the number of earned degrees in all the natural sciences.

**Estimated.

[2] *AMS (MOS) Subject Classification Scheme* (1970). Mathematical Offprint Service, American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02904.

[3] *Notices*, vol. 13, no. 5, August, 1966, pp. 579, 636–15.

The source of Table 1.1 also gives projections through 1980–81, based on past trends. Since, however, there is considerable evidence that an employment plateau has been reached in mathematics and many of the other sciences, it is unlikely that the trend will continue, at least in the near future. Projections, therefore, are not given in Table 1.1, for they might be misleading.

A related qualitative trend, nevertheless, is clearly discernible. Mathematics, which traditionally has been applied chiefly in the fields of physical science and engineering, is increasingly being utilized in the biological, social, and managerial sciences. These new applications, in turn, are leading to the creation of new mathematical techniques. Though this trend is as yet hard to document statistically, its implications for mathematics and many other fields of human endeavor are extremely significant. Though the chapter bibliography includes a few illustrative references, many articles demand a great deal of knowledge of mathematics or of the field of applications.

A "straw in the wind" was the Conference on the Application of Undergraduate Mathematics in the Engineering, Life, Managerial and Social Sciences.[4] A dozen speakers, each with advanced training in both mathematics and another specialty, presented examples of applications of mathematics in political science, psychology, sociology, physiology and medicine, ecology, computer science, industrial management, national security, and national educational policy. Even so, many applications were not included: a few are architecture, econometrics, communications and languages, and the catchall, operations research. The scope of mathematics used, moreover, was largely confined to what is expected of an undergraduate mathematics major. Whether or not one classifies these applications strictly as mathematics, it is clear that a professional working in almost any field will find a substantial background in mathematics a decided advantage, and, more and more frequently, a requirement.

We trust that you are by now properly convinced, if not overwhelmed, by the evidence that mathematics is a living, growing force in our culture. If your aim, moreover, is really to know "modern mathematics," you are attempting the impossible: even the fully trained mathematician cannot keep up with developments in all fields of mathematics, to say nothing of their applications. Fortunately, however, even a person with a limited background in the subject can profitably explore selected elementary portions of relatively recent mathematics, especially if the new material is well organized within itself and is related to his or her previous knowledge and experience. Accordingly, the tasks of this book are to weave strands of the "new" and the "old" into a coherent whole, and, more than this, to develop a modern point of view and an appreciation of the nature of mathematics today.

Suggestion for Investigation

1. It used to be assumed that the principal vocational opportunity for college mathematics majors was teaching at the secondary or college levels. While

[4]This conference, partially supported by the National Science Foundation, was held at Georgia Institute of Technology June 13–15, 1973.

this assumption has never been strictly true, it is certainly false now. To check on the current career opportunities in mathematics, consult references such as the frequently revised pamphlet, *Professional Opportunities in Mathematics*.[5]

Section 1.2 What Are the Characteristics of Mathematics?

If a person is asked, "What is mathematics?," the answer will doubtless be influenced strongly by the number and quality of the mathematics courses in his or her formal education and their applications to other courses or to his or her occupation. For example, one whose chief experience has been three or four years of high school mathematics, plus a physical science, and perhaps a course involving simple statistics, may think of mathematics primarily as a study of quantity and measurement. Numbers, and symbols which represent numbers (e.g., x, y, z), are, to be sure, important components of mathematics; and the theory and techniques of manipulating them comprise a large part of most elementary mathematics courses.

An obvious next step in answering the question "What is mathematics?" is to consult a dictionary. One dictionary in wide use defines mathematics as "a science that deals with the relationship and symbolism of numbers and magnitudes and that includes quantitative operations and the solution of quantitative problems."[6] This quotation neatly supports the ideas of the preceding paragraph. It is clear, then, that we should list as *one* characteristic

1. Mathematics includes the study of number and quantity.

Persons with a high school background, however, know that much of the geometry they studied is concerned more with shape and spatial relations than with number. Although numbers are used in plane geometry for simple counting and measuring, the fundamental ideas of congruence and similarity, as usually developed, are nonnumerical. Other nonnumerical properties are symmetry, order (one point between two others on a line), and incidence (intersection of lines and/or planes). At a more advanced level, the field of projective geometry was elaborated in the nineteenth century with little reference to number or quantity. Certainly, then, another characteristic is that

2. Mathematics includes the study of spatial configurations and properties.

A popular conception, noted earlier, is that mathematics is a set of techniques and rules of computation, or, worse still, a "bag of tricks." One important objective of this chapter is to disabuse you of this narrow view. It does not follow, however, that techniques are to be ignored. On the contrary, the

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[5] *Professional Opportunities in Mathematics*, 8th ed., 1971. Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D.C. 20036.

[6] *Webster's Third New International Dictionary* (Springfield, Mass.: G. & C. Merriam Co., 1961), p. 1393.

dramatic rise of computer science owes its origin to a combination of mathematics and engineering; and computers provide a powerful tool for hitherto impractical research. It is well to recognize, however, that the computational aspect of mathematics leads to theoretical questions, such as the efficiency of a given computer program in terms of accuracy, speed, and generality, and to the development of new methods of calculation — usually approximate in nature — which exploit the advantages of the electronic computer. With this understanding, it is only fair to claim as another characteristic that

3. Mathematics includes techniques for computing and manipulating symbols.

It is often said that mathematics is a language. Obviously it has a set of symbols that can be combined according to certain rules of syntax to form expressions or sentences. The use of these symbols really begins in the early elementary grades, or before, with the recognition and writing of Hindu-Arabic and Roman numerals. On graduation from high school a student knows many additional symbols and a considerable number of rules for combining them. In solving "word problems" in algebra, he or she has even had experience in "translating" English sentences into algebraic equations. Much of the symbolism, moreover, is universally employed by mathematicians, whatever their native tongues; in this sense mathematics is a kind of international language. Thus another characteristic is that

4. Mathematics constitutes a symbolic language.

In the teaching of high school geometry — and increasingly also arithmetic and algebra — logical reasoning is stressed. Reasoning, as distinct from mere guesswork or a rough sketch or drawing, is first used informally to organize a set of statements into a coherent whole. Next, the system is formalized by being cast in a deductive mold: assumptions (axioms or postulates) and definitions are clearly stated, and from them, by the application of logical rules, theorems are proved. Increasingly the emphasis is on reasoning with abstract concepts, such as "ideal" triangles and circles, rather than on applications to approximately triangular or circular objects, such as house roofs or wheels. So another characteristic is that

5. Mathematics utilizes the study and application of reasoning to both abstract and concrete objects.

From a slightly different standpoint, mathematics can be considered as a study of abstract structures. In high school geometry, for example, instead of emphasizing the processes of reasoning and the methods of proof, one can focus on the result: the abstract system of Euclidean plane geometry. In a similar fashion, one can speak of the system of ordinary arithmetic, in which abstract numbers like 2 and 3 can be combined by an abstract operation called addition to produce the number 5. In fact, a substantial portion of this text will examine this very system and its extensions to include negative integers, fractions, and other kinds of numbers. Moreover, by changing one or several fundamental assumptions of an abstract system, mathematicians have created different geometrical and algebraic systems, some of which turn out to have unexpected beauty or utility. Hence,

6. *Mathematics includes the study of abstract systems.*

One aspect of mathematics which, though present from the beginning of history, has become increasingly explicit in modern times is its role as a reservoir of theoretical structures (or *models*, as they are often called) which can be applied in a variety of practical situations. The Euclidean geometry of high school, for example, provides an accurate description of most of the spatial phenomena of the world in which we live, and is therefore a useful model. For instance, if a surveyor stakes out a large triangle on a flat piece of ground and measures the angles at the three stakes, he finds that the sum of the angles is 180°, within the limits of accuracy of his measuring instruments. As a trivial example, the abstract arithmetic sum $15 + 25 = 40$, when applied to the transaction of buying a 15¢ candy bar and a 25¢ drink at a snack bar, correctly predicts that the cost of the two articles together is 40¢. So we say that the geometric and arithmetic models apply in the respective physical situations just described. (Curiously, these models, familiar though they are, may fail to apply under other circumstances. Euclidean geometry, for instance, is not an accurate model for describing the dynamics of the atom. And even the arithmetic sentence $15 + 25 = 40$ does not correctly predict the volume of liquid obtained by mixing 15 cm^3 of ethyl alcohol and 25 cm^3 of water; the actual volume is measurably less than 40 cm^3.)

Historically, much of mathematics has been created specifically to solve important concrete physical problems. In a number of cases, however, the order has been reversed: an abstract system created for theoretical purposes, with no thought of applications, was later found to fit a set of experimental data. Often, indeed, the order has been mixed: a practical problem would suggest a model which, in turn, would be developed theoretically far beyond the scope of the original problem; then the extended model would find application in an entirely different physical situation, which would again inspire another abstract model, etc.[7] A most important characteristic, therefore, is that

7. *Mathematics provides abstract models which can be applied to concrete phenomena.*

Suggestions for Investigation

1. Look up the word *mathematics* in several dictionaries. Do you consider the definitions adequate? Why or why not? (You may learn more about dic-

[7] As documented near the end of Section 1.1, these problems and situations are not restricted to the physical sciences, but occur also in many fields formerly not associated with mathematics.